Pólya's Recurrence Theorem

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Abstract

Pólya's recurrence theorem states: a simple random walk on a d-dimensional lattice is recurrent for $d = 1, 2$ and transient for $d > 2$. In this paper we discuss proof for this theorem by formulating the problem as an electric circuit problem and using Rayleigh's short-cut method from classical theory of electricity.

1 Introduction

The recurrence problem first occurred to Pólya in early 1900s when he was taking a walk in a park and he crossed the same couple quite often in that park, even though it seemed to him that they both were taking random walks [4]. George Pólya investigated this problem on infinite graphs and presented his famous recurrence theorem in 1921 [3]. Peter Doyle [1], in his dissertation, showed how to apply Rayleigh's short-cut method from classical theory of electricity to prove Pólya's recurrence theorem. In this paper we discuss Doyle's proof for Pólya's theorem.

1.1 Problem

Figure 1 shows the type of infinite graphs that Pólya considered (we will refer to them as 'lattices'). Pólya's recurrence theorem states that a random walk is recurrent in 1 and 2-dimensional lattices and it is transient for lattices with more than 2 dimension. Pólya defined a random walk as recurrent if the walker passes through every single point on a lattice with probability one, otherwise the walk is transient. Doyle defined a random walk as recurrent if the walker returns to its starting point with probability one, and if there is a positive probability that the walker may not return to its starting point then the random walk is considered transient. As Doyle notes, these two recurrent walk definitions are essentially same, because if a walk is recurrent as per Pólya's definition then it is recurrent as per Doyle's definition and also the other way around.

In this paper we use Doyle's definition of recurrence as it helps us frame our recurrence problem as an electric circuit problem. Formalizing Doyle's recurrence definition: if we denote the probability that the walker never returns to its starting point by p_{escape}, then we can say a walk is recurrent if $p_{escape} = 0$ and a walk is transient if $p_{escape} > 0$.

1.2 Intuition

On a 1-D lattice after a random walker takes the first step he has a probability of $1/2$ of returning to its starting point in second step. Whereas in a 2-D lattice, the probability that he will return to the starting point in second step is $1/4$, and for 3-D dimension it is $1/6$. This is because the number of paths going away from the starting point that the walker can take increase as dimension of the lattice increases. Thus, we can see that as the dimension of a lattice increases it becomes harder for a random walker to return to its starting point. It is more intuitive if we use Pólya's recurrence definition: as dimension increases the number of points on the lattice increase, and naturally it becomes harder for a walker to cover every single point. It

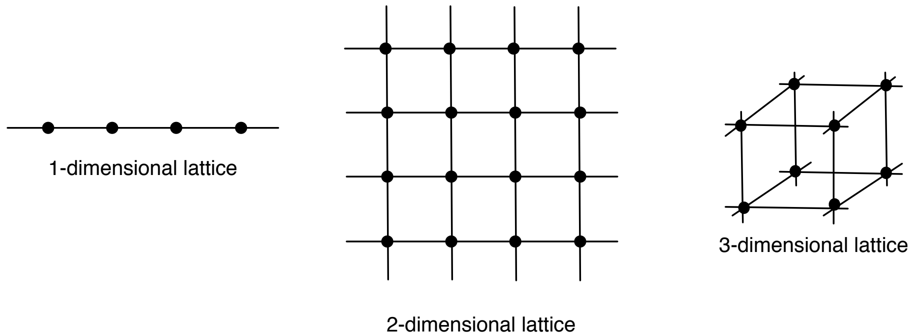

Figure 1: Infinite lattices in 1, 2, and 3 dimension

is, however, not intuitive as to why the walk becomes transient for $d > 2$ (and say, not for $d > 4$). We discuss this cutoff in Section 4.

2 Preliminaries

For our proof we need a relation between effective resistance and escape probability. We can represent a simple random walk on a graph shown in Figure 2a) that starts at a and ends in b with an electric circuit shown in Figure 2b) where each edge has a 1-Ω resistor and a 1-V battery is applied between the starting point and the end points of the walk (point a and b in the Figure).

When we represent a simple random walk as an electric circuit with 1-V battery we can interpret the voltages at each point as the probability that the walker starting at that point reaches a point of 1-V before a point of 0-V. For example, in the above Figure, V_c, the voltage at a point c represents the probability that a walker at point c will reach point a before b. We will not go into details of this proof here; we point the reader to Doyle and Snell [2] for details on voltage interpretation of probabilities.

We can express p_{escape}, the probability the walker starting at a reaches b as,

$$p_{escape} = 1 - p_{return} = 1 - \sum_x P_{ax} p_x \tag{1}$$

where p_{return} is the probability that walker starting at a returns to a, which is equal to the sum of probabilities that he reaches a before b from x (p_x) given that he has reached x from a (P_{ax}).

We know that the effective resistance between two points is the ratio of total voltage between them and the total current flowing between them, i.e., the effective resistance between a and b is $R_{eff} = V_{ab}/I_{ab}$. In our circuit all the current flows out of a and into b, thus $I_{ab} = I_a$, the total current flowing out of a.

$$\begin{aligned} I_a &= \sum_x (V_a - V_x) C_{ax} = V_a \sum_x C_{ax} - \frac{C_a}{C_a} \sum_x C_{ax} V_x = C_a - C_a \sum_x \frac{C_{ax}}{C_a} V_x = C_a (1 - \sum_x P_{ax} V_x) \\ \Rightarrow I_a &= C_a p_{escape} \quad \text{Using } p_x = V_x \text{ and Equation 1} \end{aligned} \tag{2}$$

where C_{ax} is the conductance of the edge ax, and C_a is total conductance from a ($C_a = \sum_x C_{ax}$). Thus, we have:

$$\begin{aligned} R_{eff} &= V_{ab}/I_{ab} = \frac{V_{ab}}{C_a p_{escape}} \\ p_{escape} &= \frac{V_{ab}}{C_a R_{eff}} \end{aligned} \tag{3}$$

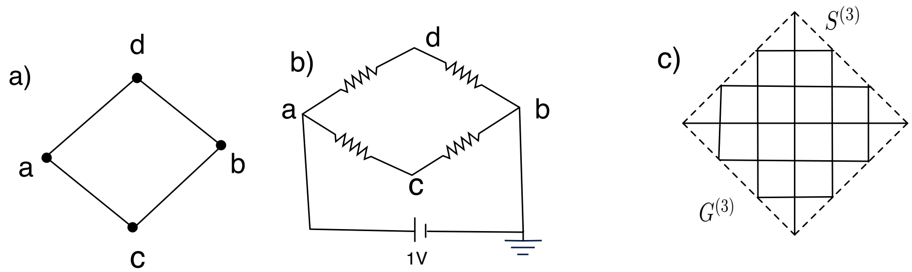

Figure 2: a) Simple random that starts in a and ends in x, b) Electric circuit representing simple random walk in a); each resistor is 1-Ω. c) $G^{(3)}$ obtained from a 2-D lattice.

3 Proof

We first convert an infinite lattice to a finite graph $G^{(r)}$ of radius r as follows: we mark the starting point (of a random walk) on Z^d as the origin 0 and throw away all the edges in the lattice that are more than r edges away (by shortest path) from 0. The left-over graph is $G^{(r)}$ and we denote all the extreme points of the graph (i.e., points that are r edges away from 0, again, by shortest path) as $S^{(r)}$. As we increase r to infinity, $G^{(r)}$ becomes an infinite graph. A random walk on $G^{(r)}$ begins at 0 and end when the walker reaches a point in $S^{(r)}$. Figure 2c shows $G^{(r)}$ on Z^2 for $r = 3$.

We denote $p^{(r)}_{escape}$ as the escape probability of $G^{(r)}$, i.e., the probability that a random walk on $G^{(r)}$, starting at 0 (we will refer to this as the 'origin'), reaches $S^{(r)}$ before returning to 0. And we denote $p_{escape} = \lim_{r\to\infty} p^{(r)}_{escape}$ as the escape probability of the infinite graph.

We can convert $G^{(r)}$ to an equivalent electric circuit as described in Section 2 and attaching a 1-V battery with the origin at 1V and all points in $S^{(r)}$ at 0V. For such an electric circuit from Equation 3 we have:

$$p^{(r)}_{escape} = \frac{1}{C_a R^{(r)}_{eff}} = \frac{1}{2dR^{(r)}_{eff}} \tag{4}$$

where C_a is the total conductance at origin and it equal to the degree of origin, $2d$. Taking limit $r \to \infty$ on both sides we get the relation between escape probability for an infinite graph and effective resistance of an infinite circuit.

$$p_{escape} = \frac{1}{2dR_{eff}} \tag{5}$$

From this equation we have: $p_{escape} = 0$ if and only if $R_{eff} = \infty$. Thus, it suffices to show that if $R_{eff} = \infty$ for an infinite lattice, a random walk on it is recurrent.

3.1 1-D and 2-D

An infinite 1-D lattice circuit is formed by attaching resisters in series forming an infinite chain. It is easy to see that for such a circuit the effective resistance to infinity is infinity. This was easy to see because of rotational symmetric. 2-D (and higher dimension) lattices, however, lack rotational symmetric. So it is not so obvious to deduce whether R_{eff} is infinity for $d > 2$.

For 2-D lattice we use Rayleigh's shorting law, which says that if we short two points in an electric circuit then the effective resistance between two given points on that circuit can only decrease. We short the edges for a 2-D lattice as shown in Figure 3a. Now we can easily compute the resistance to infinity as follows. Resistors between two levels (marked 0, 1, 2, 3, ... in the figure) are in parallel and would give one resistor of value $\frac{1}{8n+4}\Omega$ where n is the value of the smaller level. Thus, the resistance to infinity is $\sum_{n=0}^{\infty} \frac{1}{8n+4} = \infty$. Since shorting can only decrease the effective resistance, we conclude that actual resistance to infinity of a 2-D lattice is ∞.

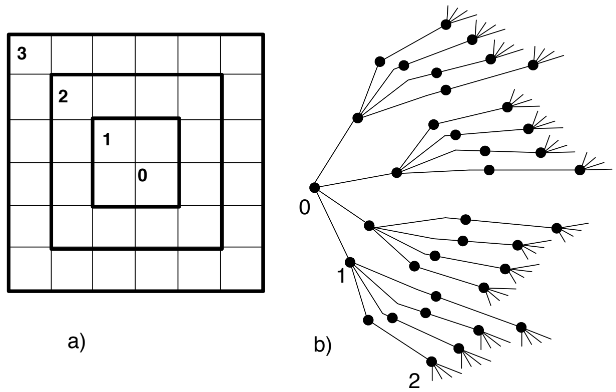

Figure 3: a) Shorting along bold edges in a 2D lattice. b) NT_3

3.2 3-D

Recall, we need to prove that a walk is transient on 3D lattice, i.e., $p_{escape} > 0 \Rightarrow R_{eff} < \infty$. Following our approach for 2D we need to obtain a nice graph from 3D lattice that we can analyze and compute R_{eff}. We cannot use Rayleigh's shorting law because we have to prove an upper bound on R_{eff}. Instead we use Rayleigh's cutting law, which states that if we cut an edge in an electric circuit, then the effective resistance between two given points in that circuit can only increase. We cut edges in 3D lattice to get a nice graph, and if we get this R_{eff} of our graph turns out to be less than infinity then we are have our proof (because the R_{eff} of the original lattice can only be less than the R_{eff} of our graph, by Rayleigh's cutting law).

Trees are easy to analyze and a full binary tree has resistance to infinity, $R_{eff} = 1$. But unfortunately we cannot obtain a binary tree from a 3D lattice (or in other words we cannot fit a binary tree in a 3D lattice) because binary trees grow too fast. If we consider a 'ball' of radius r, in a 3D lattice the total number of nodes inside such a ball is r^d (roughly speaking) whereas for a tree this number is $2^{r+1} - 1$. If we consider how quickly a 3D lattice and a tree grow, roughly speaking, in a 3D lattice every time the radius becomes doubles the size of the ball (surface area of ball) in lattice quadruples, whereas for a tree every time radius increases by 1 the size of sphere doubles. So we need to modify the tree so that it grows as slowly as the 3D lattice. We modify the tree as shown in Figure 3b. Instead of dividing into branches at each edge, we allow branching in the tree when the radius become double, and since in 3D lattice size of sphere quadruples, we make four branches. We call this tree as NT_3. Computing R_{eff} for NT_3 is easy. By symmetry the voltage at each level in the tree is same, so we can short those points and we get the resistance to infinity for NT_3 as $R_{eff} = 1/2$.

When we try embed NT_3 into a 3D lattice, the resulting tree that we actually embed turns out to be of $NT_{2.5849}$, i.e., of dimension 2.5849. We say it is of dimension 2.5849 because the number of nodes in a ball of radius r for this tree is proportional to $r^{2.5849}$ (for a 3D ball of radius r the number of nodes inside it would be proportional to r^3). The R_{eff} for $NT_{2.5849}$ is 1. So what we have now is a graph obtained from a 3D lattice whose resistance to infinity, which is finite, is equal to or larger than that of 3D lattice. Thus, we can say that resistance to infinity for a infinite 3D lattice is finite, and hence a walk on a 3D lattice is transient.

For details on how to embed a tree in a lattice and details on computing resistance to infinity for tree we refer readers to Doyle and Snell [2].

4 Discussion

4.1 Other proofs

We will briefly mention some other existing proofs for Pólya's theorem. The classical proof uses the fact that if a walk is recurrent the walker will keep returning to the origin (his starting point) and hence the expected number of times that walker will return to the origin is infinite for a recurrent walk and less than infinity for a transient walk. The proof concludes by showing that this expected number of returns is indeed infinite for walks in 1 and 2-D, and less than infinity for walk in 3-D. One must see this classical proof to appreciate the elegance of the Doyle's proof using electric circuits.

Tetali [5] formulates the problem into an electric circuit problem, but slightly differently. We applied a voltage of 1V to our circuit, whereas Tetali passes a flow of 1-amp current in the circuit. This subtle difference only changes how you relate effective resistance to a recurrent walk (i.e., how you show that a walk is recurrent if effective resistance is infinity). After that the proof for Pólya's theorem would be same.

Doyle and Snell [2] give another proof for Pólya's theorem using flows. The idea is to inject a flow at the origin of the lattice and to determine if the flow to infinity has a finite dissipation. If flow has finite dissipation the walk on the lattice is transient.

4.2 Why $d > 2$?

Although this is not a serious question, a reader may wonder why does a walk become transient above 2 dimension and not after 4 dimension, for example. Unfortunately there is not good answer to this question. Doyle's [1] explanation may satisfy some readers: if we convert this problem to a continuous problem by replacing our d-dimension resistor lattice with a resistive medium filling and the try to compute the effective resistance to infinity of this medium we would get an expression similar to: $R_\infty \int_a^\infty \frac{dr}{r^{d-1}}$, and the cutoff for this integral if $d > 2$. This justification does not really answer the question. We think a more intuitive answer remains to be found for this question to satisfy readers' curiosity.

4.3 Connecting dots

We mentioned that Pólya thought about the recurrence problem because he was meeting the same couple often during his random walk. A reader may wonder how does this theorem answer that question. Using Pólya's theorem one can show that the two walkers taking random walks on a 2D lattice would certainly meet. Here is a hint for the readers: the two random walks can be combined into one random walk with two steps each time and when the combined random walk crosses to its origin the two walkers meet.

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