Abstract

Let $G_{\lambda}$ be the graph whose vertices are points of a planar Poisson process of density $\lambda$, with vertices adjacent if they are within distance 1. A “fire” begins at some vertex and spreads to all neighbors in discrete steps; in the meantime $f$ vertices can be deleted at each time-step. Let $f_{\lambda}$ be the least $f$ such that, with probability 1, any fire on $G_{\lambda}$ can be stopped in finite time. We show that $f_{\lambda}$ is bounded between two linear functions of $\lambda$. The lower bound makes use of a new result concerning oriented percolation in the plane.

Keywords: firefighting, random geometric graph, orientated percolation.

1 Introduction

The Firefighter Problem was introduced by Hartnell [7] in 1995. A fire starts at a vertex in a graph $G$ at time 0, and spreads to all neighboring vertices in successive discrete time steps. Between each of these epochs, $f$ vertices are “protected” (equivalently, removed), where $f$ is some fixed positive integer representing the number of firefighters. When a vertex is burning or has been protected, it remains in that state. The process terminates when the fire can not spread any longer; the objective, when $G$ is infinite, is to determine the minimum number of firefighters needed to stop any fire in finite time.

For a survey of results, see [3]. So far, most of the work done on infinite graphs has been on plane lattices such as the square, triangular, and hexagonal lattice [4, 9, 10]. For fighting fires in a forest, as opposed to an orchard, it is natural instead to consider graphs whose vertices are distributed randomly on the Euclidean plane; vertices will be adjacent if they are close enough to permit the fire to spread from one to another. Taking the threshold distance to be one leads to the random geometric graphs studied here.

Note that the firefighter model is designed for graphs whose growth rate is quadratic; that is, whose neighborhoods of radius $r$ have boundaries of size roughly linear in $r$, as is typical for planar graphs. One would expect that the required number of firefighters for such a graph would be more or less proportional to its average degree, provided that the best firefighting strategy is the obvious one of surrounding the fire with protected vertices. We will show that this is indeed the case for random geometric graphs.
2 Notation and Preliminaries

Definition 2.1. A graph $G$ is geometric if its vertices are points of the plane $\mathbb{R}^2$, with $u,v$ adjacent just when their Euclidean distance $\rho(u,v)$ is less than one.

In Figure 1, unit-radius circles are drawn around each vertex to indicate which other vertices are adjacent.

Definition 2.2. A graph $G_\lambda$ is said to be a random geometric graph if its vertices are the points of a homogeneous Poisson process $\mathcal{P}_\lambda \subset \mathbb{R}^2$ of density $\lambda$.

If $S$ is a finite set of vertices in $G_\lambda$, we denote by $f(S)$ the least number of firefighters needed to stop a fire that begins (or has expanded to, by the time firefighting commences) with the vertices in $S$. The maximum $f(G_\lambda)$ of $f(S)$ over all finite vertex sets $S$ of $G_\lambda$ exists (as we shall see) for nearly all choices of $G_\lambda$: in fact, there is a least finite value, which we denote by $f_\lambda$, representing the least number of firefighters needed in order to stop any fire in a random $G_\lambda$, with probability one.

If $G_\lambda$ has no infinite component, then no firefighters are needed and thus $f(G_\lambda) = 0$. If $\lambda_c$ is the critical density for “lily-pad” percolation (see, e.g., [2]) then this will be the case with probability 1 for $\lambda < \lambda_c$, so in that case $f(\lambda) = 0$. One cannot thus expect to bound $f_\lambda$ by positive multiples of $\lambda$, but we will show:

Theorem 2.3. There exist positive constants $c_1$, $c_2$, and $c_3$ such that

$$c_2 \lambda - c_3 \leq f_\lambda \leq c_1 \lambda.$$ 

3 Proof of the Main Result

3.1 An Upper Bound on $f_\lambda$

Finding an upper bound is relatively easy.

Theorem 3.1. $f_\lambda \leq [2\pi \lambda]$.

Proof. We must show that with probability 1, any fire starting at finite vertex set $S$ of $G_\lambda$ can be stopped if we have $[2\pi \lambda]$ firefighters. To do this, we look for an annulus $A$ of annular width 1 that surrounds $S$ and does not contain more than its expected number of vertices. All vertices in $A$ are protected before the fire reaches the annulus, and no further progress by the fire is then possible.

In order to prove this proposition, we need some preliminary material.
Table 1: \( g(k) \) for \( k = 1, 2, \ldots, 8 \)

<table>
<thead>
<tr>
<th>( k )</th>
<th>( g(k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( e^{-1} \approx 0.367879 )</td>
</tr>
<tr>
<td>2</td>
<td>0.406006</td>
</tr>
<tr>
<td>3</td>
<td>0.42319</td>
</tr>
<tr>
<td>4</td>
<td>0.43347</td>
</tr>
<tr>
<td>5</td>
<td>0.440493</td>
</tr>
<tr>
<td>6</td>
<td>0.44568</td>
</tr>
<tr>
<td>7</td>
<td>0.449711</td>
</tr>
<tr>
<td>8</td>
<td>0.452961</td>
</tr>
</tbody>
</table>

Lemma 3.2. Let \( k \) be a fixed positive integer. For \( x > 0 \), the function \( g_k(x) = e^{-x} \sum_{i=0}^{k-1} \frac{x^i}{i!} \) is a strictly decreasing function of \( x \).

Proof. We know that \( g_k(x) \) is a continuous function. Its derivative is

\[
-e^{-x} \sum_{i=0}^{k-1} \frac{x^i}{i!} + e^{-x} \sum_{i=0}^{k-2} \frac{x^i}{i!}
\]

which is less than zero for \( x > 0 \).

\( \square \)

Remark 3.3. In particular, for \((k - 1) < \mu \leq k\), \( g_k(\mu) \) is strictly decreasing.

Lemma 3.4. For a positive integer \( k \), \( g(k) = e^{-k} \sum_{i=0}^{k-1} \frac{k^i}{i!} \geq e^{-1} \). For \( k > 1 \), this inequality is strict.

Proof. We know from [11] that the median of \( \text{Bi}(n, p) \), the Binomial distribution, is either \([np]\) or \([np]\). On the other hand, we know that the Poisson distribution of density \( \lambda \) is the limit of \( \text{Bi}(n, p) \) as \( n \to \infty \), where \( \lambda = np \) is fixed. As a result, if \( X \) is a Poisson random variable with mean \( k \), then its median is also equal to \( k \). This means \( P(X_k < k) \) and \( P(X_k > k) \) are both at most \( \frac{1}{2} \); therefore, \( P(X_k < k) \) is at least

\[
\frac{1}{2} - P(X_k = k) = \frac{1}{2} - \frac{e^{-k}k^k}{k!}
\]

which we know from [12] that it is at least

\[
h(k) = \frac{1}{2} - \frac{e^{-(12k+1)^{-1}}}{\sqrt{2\pi k}} > \frac{1}{2} - \frac{1}{\sqrt{2\pi k}}.
\]

On one hand, the first eight values of \( g(k) \) are given in Table 1. On the other hand, as a continuous function of \( k \), \( h(k) \) is an increasing function with \( h(9) \approx 0.368233 \) while \( e^{-1} \approx 0.367879 \).

\( \square \)

Lemma 3.5. For a positive real \( \mu \), \( g(\mu) = e^{-\mu} \sum_{i=0}^{\lceil \mu \rceil - 1} \frac{\mu^i}{i!} \geq e^{-1} \). For \( \mu \neq 1 \), the inequality is strict.
Figure 2: A Barricade of Annular Width One to Stop the Fire

Proof. When $0 < \mu < 1$, $g(\mu) = e^{-\mu} > e^{-1}$. For $\mu \geq 1$, we know that the statement of this lemma is true for integer values of $\mu$ by Lemma 3.4. We assume that $\mu$ is a real number, and let $k = \lceil \mu \rceil$. We know that $g(\mu) = g_k(\mu)$ is a strictly decreasing function of $\mu$ in $(k-1, k]$. Consequently, $g(\mu) = e^{-\mu} \sum_{i=0}^{k-1} \frac{\mu^i}{i!} > e^{-k} \sum_{i=0}^{k-1} \frac{k^i}{i!} \geq e^{-1}$. □

Corollary 3.6. For a positive real $\mu$, $e^{-\mu} \sum_{i=0}^{\lfloor \mu \rfloor} \frac{\mu^i}{i!} > e^{-1}$.

Proof. (Proposition ??) We assume that the fire has started at a vertex $v$. For simplicity of notation, we shift the coordinates so that $v$ is the origin. We also let the fire rage for $r_0$ time units before we deploy the firefighters. First, we will divide the plane into disjoint annuli $A_r : r^2 \leq x^2 + y^2 < (r+1)^2$, where $r > r_0$ is an integer. One thing to have in mind is that, in the worst case scenario for the firefighters, the fire will spread radially, as shown in Figure 2. This means that the time for the fire to reach the inner circle of $A_r$ is at least $(r-r_0)$. Our objective is to show that for some $R$, the number of vertices in $A_R$ is less than or equal to $(R-r_0)$. If such an annulus exists, by the time the fire reaches this annulus, all the vertices in $A_R$ can be protected. Since the annular width of $A_R$ is one, the fire can not spread to the vertices that are beyond the outer circle of $A_R$.

Define $X_r$ to be the number of vertices in $A_r$. We want to show that for $r$ large enough, $\mathbb{P}(X_r \leq (r-r_0)f) > e^{-1}$. To do so, we assume that

$$r \geq \frac{\pi \lambda + r_0 f}{f - 2\pi \lambda} = m$$

or, equivalently, $(r-r_0)f \geq (2r+1)\pi \lambda$, and will prove that $\mathbb{P}(X_r \leq \lfloor(2r+1)\pi \lambda\rfloor) > e^{-1}$.

We know that the area of $A_r$ is $(2r+1)\pi$, and, as a result, $X_r$ has a Poisson distribution of density $(2r+1)\pi \lambda$. Assuming $\mu = (2r+1)\pi \lambda$ and using Corollary 3.6, we have

$$\mathbb{P}(X_r \leq \lfloor(2r+1)\pi \lambda\rfloor) = e^{-\mu} \sum_{i=0}^{\lfloor(2r+1)\pi \lambda\rfloor} \frac{\mu^i}{i!} > e^{-1}.$$ 

Since $A_r$’s are disjoint and $\mathbb{P}_\lambda$ is a Poisson process, $X_r$’s are independent. For each $r \geq m$, define the Bernoulli random variable $Y_r$ such that $\mathbb{P}(Y_r = 1) = \mathbb{P}(X_r \leq (r-r_0)f)$. Since $X_r$’s are independent, so are $Y_r$’s. Based on the above observation, the probability of success $Y_r = 1$ is greater than $e^{-1}$. It follows that for some $R \geq m$, $Y_R = 1$. □
In this section, we will prove the following theorem:

**Theorem 3.7.** For all $\lambda$, if $\lambda$ is greater than the maximum of $75$ and

$$
\frac{1}{8}(15f + 60 + 80 \ln(3) + \sqrt{(15f + 60 + 80 \ln(3))^2 - \frac{1}{4}(5f + 60)^2}),
$$

then $f$ is not sufficient for $\vec{G}_\lambda$.

With regard to **Theorem 3.1**, it can be argued that there might be a better strategy to contain the fire that involves slowing down the fire by protecting some key vertices while the rest of the firefighters try to surround the fire. As it turns out, when $\lambda$ is large enough, this strategy does not work even in the case of a weakened fire that spreads in a particular direction, and, as a result, can not take any circuitous paths. To prove this statement, we use techniques from orientated percolation that allow us to demonstrate that, even in the case of an oriented fire, the best one can hope for is to stop the fire by building a barricade of annular width one.

In the case that $\vec{G}$ is an oriented graph, the fire spreads only through the out-edges of a vertex. Suppose $\vec{Z}^2$ is the oriented square lattice in which the horizontal and the vertical edges are oriented from left to right and bottom to top, respectively, as shown in Figure 3.2.

Previously, we assumed that $f$ is a fixed positive integer, but in order to prove the next few results, we assume that $f$ is a fraction $\frac{a}{b}$ with $0 < a < b$. What this means is that we deploy $a$ firefighters after every $b$ time units.

Let a fire start at $v$ in $\vec{Z}^2$, and let us assume that the vertices in $\vec{Z}^2$ are open (vulnerable to the fire), with probability $p$. When a vertex is not open, we call it closed, and, by default, a closed vertex is immune from the fire. Moreover, an open vertex will be closed when it is protected by a firefighter.

**Proposition 3.8.** Let $f \in (0, \frac{1}{2})$ be a fixed fraction. If $p > 1 - 9^{-\left(\frac{4}{3} - f\right)^{-1}}$, then $f$ is not sufficient for $\vec{Z}^2$, with positive probability.

Before we start with the proof of **Proposition 3.8**, we need to state the following theorem whose proof can be found in [8]:

![Figure 3: $\vec{Z}^2$](image-url)
Figure 4: A Blocking Cycle

**Theorem 3.9.** (Chernoff-Hoeffding) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with $X_i \in \{0, 1\}$ and $p = \mathbb{E}[X_i]$ for all $i$. Then for $\epsilon > 0$,

\[
\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i \geq p + \epsilon\right) \leq \left(\frac{p}{p+\epsilon}\right)^{p+\epsilon}\left(\frac{1-p}{1-p-\epsilon}\right)^{1-p-\epsilon}^n
\]  

and

\[
\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i \leq p - \epsilon\right) \leq \left(\frac{p}{p-\epsilon}\right)^{p-\epsilon}\left(\frac{1-p}{1-p+\epsilon}\right)^{1-p+\epsilon}^n.
\]

If we define

\[
\Psi(x, y) = x \ln\left(\frac{x}{y}\right) + (1-x) \ln\left(\frac{1-x}{1-y}\right),
\]

then the right-hand side of (1) and (2) simplify to $\exp(-n\Psi(p \pm \epsilon, p))$. Moreover, we know that for $x \in (0, \frac{1}{2})$ fixed and $y \in [x, 1)$, $\Psi(x, y)$ is a strictly increasing, nonnegative function. It follows that $\exp(-\Psi(x, y)) \leq 1$.

**Proof.** (Proposition 3.8) Let $C_v$ be the set of all vertices that can be reached from $v$ by open paths. If we do not deploy any firefighters, the fire will reach these vertices eventually. Suppose that the fire is contained, and, as a result, $C_v$ is finite. Consequently, there exists an external boundary $\partial C_v^\infty$ blocking the fire from getting from the interior to the exterior of this boundary.

Let $\mathbb{Z}^{2*}$ be the face-to-vertex dual of $\mathbb{Z}^2$, and let $S$ be any cycle of length $2l$ in $\mathbb{Z}^{2*}$ that we traverse counterclockwise. Suppose $S$ is a blocking cycle, as shown in Figure 3.2. In this figure, “○” denotes a closed vertex. There are four observations to make:

1. The edges in $S$ that are blocking the fire are those who are oriented upward if vertical, or leftward if horizontal. This is due to the way the fire spreads in $\mathbb{Z}^{2*}$. 

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2. The number of upward and leftward edges in \( S \) is equal to the number of downward and rightward edges, respectively. This means that, as we traverse \( S \), we are making a total of \( l \) upward and leftward steps.

3. For \( S \) to be blocking, any upward or leftward edge in \( S \) must have a closed vertex to its right.

4. Any vertex can be to the right of at most one upward and at most one leftward step.

Putting these observations together, for \( S \) to be blocking, there are at least \( \frac{l}{2} \) closed vertices, when the fire is contained. Moreover, the graph distance of \( v \) to any vertex \( x \in \text{Int}(S) \) is at most \( l \) which implies that the fire must be contained by \( t = l \). So when the fire is contained, the total number of open vertices that are closed by the firefighters is at most \( fl \). As a result, at least \( \frac{l}{2} - fl \) of these vertices were closed before the fire started. Based on our observation, there are at most \( l \) vertices to the right of an upward or a leftward step in \( S \). We want to show that the probability of the event \( X \) that at most \( fl \) of these vertices were open and had to be closed by the firefighters is exponentially small. We know that before the fire starts, each vertex is open with probability \( p \), independent of other vertices, and according to a Bernoulli distribution. Let \( p = \frac{\lambda}{n} \) and \( n = l \) in (2), and we have \( \mathbb{P}(X) \leq \exp(-\Psi(f,p))l \). It follows that the probability that \( S \) is a blocking cycle is bounded from above by \( (1 - p)^{\frac{l}{2} - fl} \exp(-\Psi(f,p))l \leq (1 - p)^{\frac{l}{2} - fl} \), since \( f \in (0, 1) \).

Let \( Y_k \) be the line segment joining \( v = (i_0, j_0) \) to the lattice point \( (i_0 + k, j_0) \), and let \( Z_k \) be the number of blocking cycles around \( Y_k \). We know that the number of cycles surrounding \( Y_k \) of length \( 2l \) is bounded from above by \( 4 \times 3^{2l-2} \), and each cycle is blocking, with probability at most \( (1 - p)^{\frac{l}{2} - fl} \). As a result, for \( p > 1 - 9^{-\left(\frac{1}{2} - f\right)^{-1}} \),

\[
\mathbb{E}[Z_k] \leq \sum_{l \geq k+2} l(4 \times 3^{2l-2})(1 - p)^{\frac{l}{2} - fl} \leq \sum_{l \geq k+2} \frac{4}{9}l(3(1 - p)^{\frac{l}{2} - fl})^{2l}
\]

converges. For \( l \) large enough, this expected value will be less than one. Since \( Z_k \) is a counting random variable, \( \mathbb{P}(Z_k = 0) > 0 \). Define \( W_k \) to be the event that \( Z_k = 0 \), and define \( W'_k \) to be event that \( k + 1 \) vertices in \( Y_k \) are unprotected. These two events are independent, since they are disjoint. Moreover, if they both hold, there will be no blocking cycles, and, as a result, the fire will continue spreading indefinitely, with positive probability. We know that \( W_k \) and \( W'_k \) happen with positive probability.

Before we proceed, we will state a theorem regarding Poisson tail probabilities. For a proof, see [6]. Let \( \text{Po}(\lambda, k) \) denote the cumulative Poisson distribution function, and we have

\[
\text{Po}(\lambda, n) = e^{-\lambda} \sum_{k=0}^{n} \frac{\lambda^k}{k!}.
\]

Moreover, let \( \Phi(x) = \int_{-\infty}^{x} \phi(t)dt \), where \( \phi(x) = (2\pi)^{-\frac{1}{2}} \exp(-\frac{x^2}{2}) \) is the normal function.

**Theorem 3.10.** Let \( \lambda \geq 2 \) and \( a = \lfloor \lambda \rfloor \). Then for \( n \geq 2 \),

\[
\text{Po}(\lambda, a - n) \leq b_\lambda \Phi\left(\frac{n - \frac{a}{2}}{\sqrt{\lambda}}\right),
\]

where \( \Phi(x) = 1 - \Phi(x) \) and \( b_\lambda = (1 + \lambda^{-1}) \exp\left(\frac{\lambda}{8}\right) \). Provided that \( \lambda \geq 25 \), \( b_\lambda \) is less than or equal to 1.05 [5].
Proof. (Theorem 3.7) Let us assume that a fire starts at a vertex \( v \) in \( G_\lambda \). For simplicity of notation, we shift the coordinates so that \( v \) is the origin, and we will divide the plane into squares with side length \( \frac{\sqrt{5}}{5} \), as shown in Figure 3.2: For \((i, j) \in \mathbb{Z}^2\), define

\[
s_{i,j} = \{(x, y) \mid \frac{\sqrt{5}}{5} i \leq x < \frac{\sqrt{5}}{5} (i + 1), \frac{\sqrt{5}}{5} j \leq y < \frac{\sqrt{5}}{5} (j + 1)\}.
\]

Since the side of each square is \( \frac{\sqrt{5}}{5} \), every vertex in \( s_{i,j} \) is adjacent to every vertex in \( s_{i-1,j}, s_{i,j-1}, s_{i+1,j}, \) and \( s_{i,j+1} \). Let \( \mathcal{G} \) be the collection of all these squares, and define \( \mathcal{G}^\square \) to be the face-to-vertex dual of \( \mathcal{G} \). We know that each square contains no vertices in \( G_\lambda \) with probability \( e^{-\frac{\sqrt{5}}{5}} \). If a square does not contain any vertices, we call it closed; otherwise, we call it open.

Orient the edges in \( \mathcal{G}^\square \) such that the horizontal and the vertical edges are oriented rightward and upward, respectively, and denote the obtained digraph \( \bar{\mathcal{G}}^\square \). When the fire starts at \( v \), the firefighters protect \( f \) vertices in the first quadrant of \( \bar{\mathcal{G}}^\square \), and then, the fire spreads according to the orientation on \( \bar{\mathcal{G}}^\square \). We will show that for \( f \) a positive fraction, when \( \lambda \) is greater than the maximum of 75 and

\[
\frac{1}{8} \left( 60(f + 1) + 80 \ln(3) + \sqrt{(60(f + 1) + 80 \ln(3))^2 - 4(5f + 15)^2} \right),
\]

the firefighters will not be able to stop this fire which spreads through \( G_\lambda \) according to the above orientation on \( \bar{\mathcal{G}}^\square \), with high probability.
Let \( S \) be any cycle of length \( 2l \) in \( \mathcal{S} \) that we traverse counterclockwise. Suppose \( S \) is a blocking cycle, and let \( C_v \) be the set of all vertices that can be reached from \( v \) by open paths. If we do not deploy any firefighters, the fire will reach these vertices eventually. Suppose that the fire is contained, and, as a result, \( C_v \) is finite. Since \( s_{i,j}'s \) are disjoint, the probability that a square contains at least a vertex is independent of that of other squares. We call a square which satisfies this property open. An open square can be closed when all the vertices it contains are protected by the firefighters. We know from the proof of Proposition 3.8 that if \( S \) is a blocking cycle of length \( 2l \), then the fire is contained by time \( t = l \). Consequently, the total number of firefighters deployed is at most \( fl \).

We know that the area of each \( s_{i,j} \) is equal to \( \frac{1}{5} \), and, as a result, the probability distribution for the number of vertices in any square is Poisson of density \( \frac{\lambda}{5} \). Consequently, the probability distribution for the number of vertices in any collection of size \( l \) of these squares is Poisson of density \( \frac{\lambda l}{5} \). Let \( X \) be the counting random variable for the number of vertices in the squares that make \( S \) a blocking cycle. What we are after is an upper bound for \( \mathbb{P}(X \leq fl) \). We know that this probability is less than or equal to \( \text{Po}(\frac{\lambda l}{5}, fl) \). For simplicity of computations, we assume that \( \lambda \) is an integer. In order to use Theorem 3.10, we let \( n \) be equal to \( \frac{\lambda l}{5} - fl \).

Letting

\[
\frac{\lambda l - fl - 3}{2} = x_0,
\]
we have \( \mathbb{P}(X \leq fl) \leq b \lambda l \int_{x_0}^\infty \phi(t)dt \). Assuming that \( x_0 \geq 1 \), we have

\[
\int_{x_0}^\infty \phi(t)dt \leq \int_{x_0}^\infty t\phi(t)dt,
\]

since \( \phi(t) \) takes positive values. The integral on the left hand side of this inequality equals

\[
\exp\left(-\frac{(2\lambda - 5fl - 15)^2}{40\lambda l}\right).
\]

Assuming that \( \lambda \geq (5f + 15)/2 \), we have \( 0 \leq 2\lambda - 5fl - 15l \leq 2\lambda - 5fl - 15 \) and, as a result, 

\[
-(2\lambda - 5fl - 15)^2 \leq -(2\lambda - 5fl - 15l)^2.
\]

It follows that

\[
\exp\left(-\frac{(2\lambda - 5fl - 15)^2}{40\lambda l}\right) \leq \exp\left(-\frac{(2\lambda - 5fl - 15l)^2}{40\lambda l}\right).
\]

Consequently, we have

\[
\mathbb{P}(X \leq fl) \leq b \lambda l \exp\left(-\frac{l(2\lambda - 5f - 15)^2}{40\lambda}\right).
\]

The probability of \( S \) being a blocking cycle is bounded from above by

\[
\exp\left(-\frac{\lambda}{5}(\frac{l}{2} - \frac{5fl}{\lambda})\right) \cdot b \lambda l \exp\left(-\frac{l(2\lambda - 5f - 15)^2}{40\lambda}\right).
\]

The reason for the validity of the first multiplicand is that, before the fire starts, each square is closed with probability \( e^{-\frac{\lambda}{5}} \), and when the fire is contained, we know from the proof of Proposition 3.8 that there are at least \( \frac{l}{2} \) closed squares. On the other hand, there are at most \( fl \) vertices that are protected when the fire is contained and each square is expected to have \( \frac{\lambda}{5} \) vertices. Consequently, if \( N \) is the number of open squares which are closed by the
The firefighters, then $\mathbb{E}[N] \leq \frac{5f}{l}$. It follows that the probability that the rest of these squares were closed before the fire started is at most $\exp(-\frac{\lambda}{5}(\frac{l}{f} - \frac{5f}{l}))$.

Given that $v = (i_0, j_0)$, let $Y_k$ be the line segment joining $s_{i_0, j_0}$ to the lattice point $s_{i_0+k, j_0}$, and let $Z_k$ be the number of blocking cycles around $Y_k$. We know that the number of cycles surrounding $Y_k$ of length $2l$ is bounded from above by $4 \times 3^{2l-2}$. Assuming that $\lambda \geq 75 \geq \frac{75}{l}$, we have $b_{\frac{5}{l}} \leq 1.05$, and, as a result,

$$\mathbb{E}[Z_k] \leq 1.05 \sum_{l \geq k+2} l(4 \times 3^{2l-2}) \exp\left(-l\left(\frac{\lambda}{10} - f + \frac{(2\lambda - 5f - 15)^2}{40\lambda}\right)\right)$$

$$= 1.05 \times \frac{4}{9} \sum_{l \geq k+2} l \left(9 \exp\left(-\left(\frac{\lambda}{10} - f + \frac{(2\lambda - 5f - 15)^2}{40\lambda}\right)\right)\right)^l.$$

This series converges when the expression under the exponent is less than one or, equivalently, when $0 < 8\lambda^2 - (60(f + 1) + 80\ln(3))\lambda + (5f + 15)^2$. This inequality holds when

$$\lambda > \frac{1}{8}\left(60(f + 1) + 80\ln(3) + \sqrt{(60(f + 1) + 80\ln(3))^2 - 4(5f + 15)^2}\right). \quad (3)$$

For $l$ large enough, this expected value will be less than one. Since $Z_k$ is a counting random variable, then $\mathbb{P}(Z_k = 0) > 0$. Define $W_k$ to be the event that $Z_k = 0$, and define $W'_k$ be event that $k + 1$ vertices in $Y_k$ are unprotected. These two events are independent, since they are disjoint. Moreover, if they both hold, there will be no blocking cycles, and, as a result, the fire will continue spreading indefinitely, with positive probability. We know that $W_k$ and $W'_k$ happen with positive probability.

Throughout this proof, we have made some assumptions regarding $\lambda$. We assumed that $x_0 > 1$ which is equivalent to

$$\frac{\lambda l}{5} - fl - \frac{3}{2} > \sqrt{\frac{\lambda l}{5}}.$$  

This expression is quadratic in $\sqrt{\lambda l}$, and, as a result,

$$\sqrt{\lambda l} > \frac{1}{4}\left(\sqrt{5} + \sqrt{5 + 8(10fl + 15)}\right)$$

or, equivalently,

$$\lambda > \frac{1}{16l}\left(\sqrt{5} + \sqrt{5 + 8(10fl + 15)}\right)^2 = \frac{1}{8l}(65 + 40fl + \sqrt{5 + 8(10fl + 15)})$$

which is less than

$$9 + 5f + \sqrt{5 + 8(10f + 15)} \times \frac{8}{8}.$$  

This expression, as a function of $f$, is always greater than $(5f + 15)/2$. On the other hand, it is less than or equal to the expression on the right-hand side of (3).

Now think of each $s_{i,j}$ as the vertex $(i, j)$ in $\mathbb{Z}^2$. We will show that when $\lambda$ is large enough, with positive probability, the firefighters will not be able to stop the fire, even in the case that the fire is spreading according to the following orientation on $\mathcal{G}^\square$: in the first quadrant, all the vertical edges are oriented upward and the horizontal edges, rightward; the orientation in the second, the forth, and the third quadrant is the mirror image of the orientation on the first
quadrant along the $y$-axis, along the $x$-axis, and through $v$, respectively; finally, we remove the edges that cross the axes.

Let $S$ be a blocking cycle of length $2l$ in $S^*$ which we traverse counterclockwise. Let $S_i$ ($i \in \{1, 2, 3, 4\}$) be the four sections of $S$ in each quadrant. Since $S$ is a blocking cycle of length $2l$, due to the parity of the number of squares going left and right or up and down, the fire is contained by time $t = l$, and, as a result, the total number of firefighters deployed is at most $fl$. Moreover, the size of each $S_i$ is at most $l$. By the Pigeonhole Principle, one of the quadrants contains at most $fl$ protected vertices, say the first quadrant. We know that this is equivalent to defending vertices using $\frac{f}{4}$ firefighters in $\mathbb{Z}^2$; hence, when $\lambda$ is greater than

$$
\max\left\{\frac{1}{8} \left(15f + 60 + 80 \ln(3) + \sqrt{(15f + 60 + 80 \ln(3))^2 - \frac{1}{4}(5f + 60)^2}\right), 75\right\},
$$

the fire can not be contained in $\mathbb{Z}^2$, and, as a result, it can not be stopped in $G_\lambda$, with positive probability. \qed

### 3.3 Main Theorem

Recall that $f_\lambda$ is the least number of firefighters needed in order to stop any fire in $G_\lambda$, with probability one. Note that $f_\lambda = 0$, unless $G_\lambda$ has an infinite component, with positive probability. Now we will use Theorem 3.1 and Theorem 3.7 to prove the following theorem:

**Theorem 3.11.** There exist positive constants $c_1$, $c_2$, and $c_3$ such that $c_2\lambda - c_3 \leq f_\lambda \leq c_1\lambda$, that is, $f_\lambda$ is roughly linear in $\lambda$.

**Proof.** We know from Theorem 3.1 that for some $\epsilon$, $f \geq 2\pi \lambda (1 - \epsilon)$ is sufficient for $G_\lambda$. As a result, $f_\lambda \leq 2\pi \lambda$. On the other hand, we know from Theorem 3.7 that when $\lambda$ is greater than

$$
\frac{1}{8} \left(15f + 60 + 80 \ln(3) + \sqrt{(15f + 60 + 80 \ln(3))^2 - \frac{1}{4}(5f + 60)^2}\right),
$$

then $f$ is not sufficient for $G_\lambda$. Using a computer algebra software, we have

$$(15f + 60 + 80 \ln(3))^2 - \frac{1}{4}(5f + 60)^2 = \frac{25}{4} (5f + 12 + 32 \ln(3))(7f + 36 + 32 \ln(3)).$$

As a function of $f > 0$, the values of $7f + 36 + 32 \ln(3)$ are greater than those of $5f + 12 + 32 \ln(3)$. It follows that

$$
\frac{1}{8} \left(15f + 60 + 80 \ln 3 + \sqrt{\frac{25}{4} (7f + 36 + 32 \ln(3))^2}\right) = \frac{1}{8} \left(15f + 60 + 80 \ln 3 + \frac{5}{2} (7f + 36 + 32 \ln(3))\right) = \frac{1}{16} (65f + 300 + 320 \ln(3)).
$$

It follows that if

$$
\frac{16}{65} \lambda - \frac{300 + 320 \ln(3)}{65} > f,
$$

then $f$ is not sufficient for $G_\lambda$. This implies that

$$
f_\lambda \geq \frac{16}{65} \lambda - \frac{300 + 320 \ln(3)}{65}. \quad \square$$
References


