

Maximum Overhang (extended abstract)*

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Abstract

How far can a stack of n identical blocks be made to hang over the edge of a table? The question dates back to at least the middle of the 19th century and the answer to it was widely believed to be of order $\log n$. However, at SODA'06, Paterson and Zwick constructed n -block stacks with overhangs of order $n^{1/3}$. Here we complete the solution to the overhang problem, and answer Paterson and Zwick's primary open question, by showing that order $n^{1/3}$ is best possible. At the heart of the argument is a lemma (possibly of independent interest) showing that order d^3 non-adaptive coinflips are needed to propel a discrete random walk on the number line to distance d .

We note that our result is not a mainstream algorithmic result, yet it is about the solution to a discrete optimization problem. Moreover, it illustrates how methods founded in theoretical computer science can be applied to a problem that has puzzled some mathematicians and physicists for more than 150 years.

1 Introduction

The problem of stacking n blocks on a table so as to achieve maximum overhang has a long history. It appears in physics and engineering textbooks from as early as the mid 19th century (see, e.g., [P1850], [W1855], [M1907]). The problem was apparently first brought to the attention of the mathematical community in 1923 when J.G. Coffin posed it in the “Problems and Solutions” section of the American Mathematical Monthly [C1923]; no solution was presented there.

The problem recurred from time to time over subsequent years, e.g., [S1953, S1954, S1955, J1955, GS1958, E1959, G1964, G1971, A1979, D1981, GKP1988, H2005], achiev-

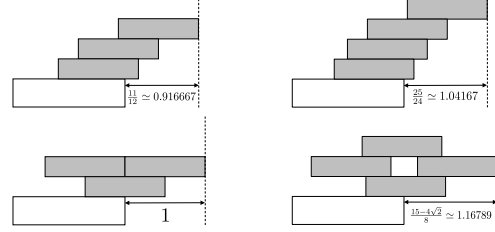


Figure 1: Optimal stacks with 3 and 4 blocks, compared to the corresponding harmonic stacks. The 4-block solution is from [A1979]. Like the harmonic stacks it can be made stable by minute displacements.

ing much added notoriety from its appearance in 1964 in Martin Gardner’s “Mathematical Games” column of *Scientific American* [G1964, G1971].

Most of the references mentioned above describe the now-classical *harmonic stacks* in which n unit-length blocks are placed one on top of the other, with the i^{th} block from the top extending by $\frac{1}{2i}$ beyond the block below it. The overhang achieved by such stacks is $\frac{1}{2}H_n = \frac{1}{2} \sum_{i=1}^n \frac{1}{i} \sim \frac{1}{2} \ln n$. The cases $n = 3$ and $n = 4$ are illustrated at the top of Figure 1 above, and the cases $n = 20$ and $n = 30$ are shown in the background of Figure 2.

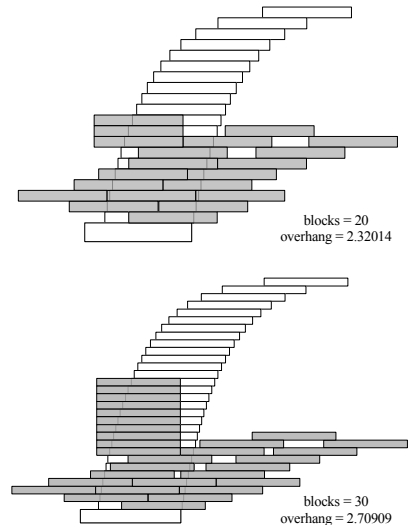


Figure 2: Optimal stacks with 20 and 30 blocks from [PZ2006] with corresponding harmonic stacks in the background.

*A full version of this paper is available as [PPTWZ2007].

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Verifying that harmonic stacks are *balanced* and can be made *stable* (see definitions in the next section) by minute displacement is an easy exercise. (This is in fact the form in which the problem appears in [P1850], [W1855], [M1907].) Harmonic stacks show that arbitrarily large overhangs can be achieved if sufficiently many blocks are available. They have been used by countless teachers as an introduction to recurrence relations, the harmonic series and simple optimization problems (see, e.g., [GKP1988]).

1.1 How far can you go? Many readers of the above mentioned references were led to believe that $\frac{1}{2}H_n$ ($\sim \frac{1}{2} \ln n$), the overhang achieved by harmonic stacks, is the *maximum* overhang that can be achieved using n blocks. This is indeed the case under the restriction, explicit or implicit in some of these references, that the blocks should be stacked in a *one-on-one* fashion, with each block resting on at most one other block. It has been known for some time, however, that larger overhangs may be obtained if the one-on-one restriction is lifted. Three blocks, for example, can easily be used to obtain an overhang of 1. Ainley [A1979] found that four blocks can be used to obtain an overhang of about 1.16789, as shown at the bottom right of Figure 1, and this is more than 10% larger than the overhang of the corresponding harmonic stack.

Now what happens when n grows large? Can general stacks, not subject to the one-on-one restriction, improve upon the overhang achieved by the harmonic stacks by more than a constant factor, or is overhang of order $\log n$ the best that can be achieved? In a recent cover article in the *American Journal of Physics*, Hall [H2005] observes that the addition of counterbalancing blocks to one-on-one stacks can double (asymptotically) the overhang obtainable by harmonic stacks. However, he then incorrectly concludes that no further improvement is possible, thus perpetuating the order $\log n$ “mythology”.

Recently, however, Paterson and Zwick [PZ2006] discovered that the modest improvements gained for small values of n by using layers with multiple blocks mushroom into an exponential improvement for large values of n , yielding overhang of order $n^{1/3}$ instead of just $\log n$. The principal construction from [PZ2006] has a parabolic shape as illustrated in Figure 3. It is a stack of “slabs” where each slab is one wider and two higher than the previous slab, and gives an overhang of $(3n/16)^{1/3} \simeq 0.57n^{1/3}$. With the removal of one block and a small adjustment in the top row, the stack can be made stable, yet achieve an overhang of $\Omega(n^{1/3})$.

1.2 Can we go further? But is $n^{1/3}$ the right answer, or is it just the start of another mythology? In their deservedly popular book *Mad About Physics* [JP2001], Jargodzki and Potter rashly claim that inverted triangles

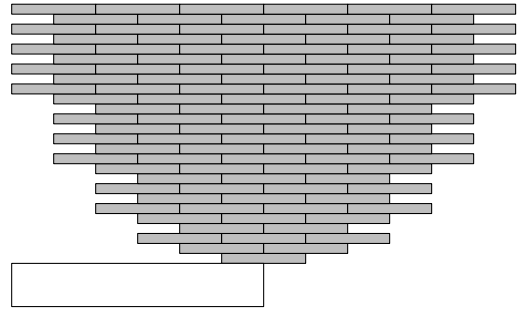


Figure 3: The $\Omega(n^{1/3})$ overhang from [PZ2006] with $n = 111$.

(such as the one shown at top left in Figure 4) are balanced. If so, they would achieve overhangs of order $n^{1/2}$. It turns out, however, that already the 3-row inverted triangle is unbalanced, and collapses as shown on the right of Figure 4, as do all larger inverted triangles.

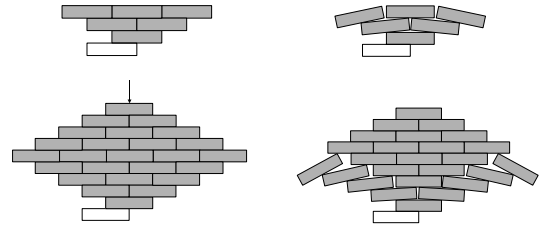


Figure 4: A 3-row inverted triangle is unbalanced, and so is a 5-diamond.

The collapse of the 3-row triangle begins with the lifting of the middle block in the top row. It is tempting to try to avoid this failure by using a diamond shape instead as illustrated in Figure 4. Diamonds were considered by Drummond [D1981], and like the inverted triangle, they would achieve an overhang of order $n^{1/2}$ though with a smaller leading constant. The stability analysis of diamonds is slightly more complicated than that of inverted triangles, but it can be shown that d -diamonds, i.e., diamonds that have d blocks in their largest row, are stable if and only if $d < 5$. In Figure 4 we see a demonstration with $d = 5$.

It is not hard to show that particular constructions like larger inverted triangles or diamonds are unstable. This concrete instability of inverted triangles and diamonds was already noted in [PZ2006]. However, this does not rule out the possibility of a smarter balanced way of stacking n blocks so as to achieve an overhang of order $n^{1/2}$, and that would be much better than the overhang of order $n^{1/3}$ achieved by Paterson and Zwick [PZ2006]. Paterson and Zwick did consider this general question, but they proved only that no larger overhang than order $n^{1/2}$ is possible. Thus their work shows that the order of the maximum overhang with n blocks lies between $n^{1/3}$ and $n^{1/2}$.

1.3 Our result. We show here that an overhang of order $n^{1/3}$ as obtained by [PZ2006] is, in fact, best possible. More specifically, we show that any n -block stack with an overhang of at least $6n^{1/3}$ is unbalanced, and must hence collapse. So the maximum overhang with n blocks is of order $n^{1/3}$.

1.4 Contents. The rest of this extended abstract is organized as follows. In the next section we present a precise mathematical definition of the overhang problem, explaining in particular when a stack of blocks is said to be *balanced* (and when it is said to be *stable*). In Section 3 we introduce a class of abstract mass movement problems and explain the connection between these problems and the overhang problem. In Section 4 we obtain bounds for mass movement problems that imply the order $n^{1/3}$ upper bound on overhang. We end in Section 5 with some concluding remarks and open problems.

2 The Model

We briefly state the mathematical definition of the overhang problem. For more details, see [PZ2006]. As in previous papers, e.g., [H2005], the overhang problem is taken here to be a two-dimensional problem: each block is represented by a frictionless rectangle whose long sides are parallel to the table. Our upper bounds apply, however, in more general settings, as will be discussed in Section 5.

2.1 Stacks. Stacks are composed of blocks that are assumed to be identical, homogeneous, frictionless rectangles of unit length, unit weight and height h . Our results here are clearly independent of h , and our figures use any convenient height. Previous authors have thought of blocks as cubes, books, coins, playing cards, etc.

A stack $\{B_1, \dots, B_n\}$ of n blocks resting on a flat table is specified by giving the coordinates (x_i, y_i) of the lower left corner of each block B_i . We assume that the upper right corner of the table is at $(0, 0)$ and that the table extends arbitrarily far to the left. Thus block B_i is identified with the box $[x_i, x_i + 1] \times [y_i, y_i + h]$ (its length aligned with the x -axis), and the table, which we conveniently denote by B_0 , with the region $(-\infty, 0] \times (-\infty, 0]$. Two blocks are allowed to touch each other, but their interiors must be disjoint.

We say that block B_i *rests on* block B_j , denoted “ B_i/B_j ”, if and only if $B_i \cap B_j \neq \emptyset$ and $y_i = y_j + h$. If $B_i \cap B_0 \neq \emptyset$, then B_i/B_0 , i.e., block B_i *rests on* the table. If B_i/B_j , we let $I_{ij} = B_i \cap B_j = [a_{ij}, b_{ij}] \times \{y_i\}$ be their *contact interval*. If $j \geq 1$, then $a_{ij} = \max\{x_i, x_j\}$ and $b_{ij} = \min\{x_i + 1, x_j + 1\}$. If $j = 0$ then $a_{i0} = x_i$ and $b_{i0} = \min\{x_i + 1, 0\}$.

The *overhang* of a stack is defined to be $\max_{i=1}^n (x_i + 1)$.

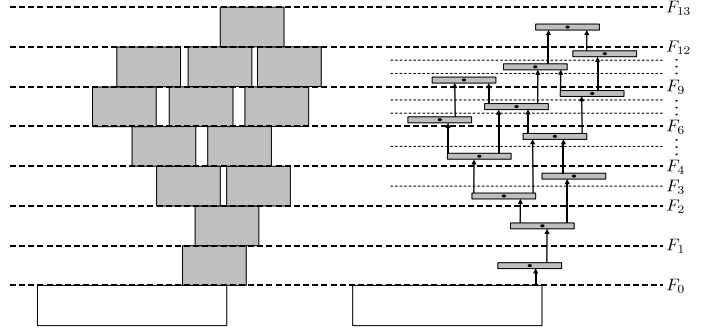


Figure 5: Balancing collections of forces within a stack.

2.2 Forces, equilibrium and balance. Let $\{B_1, \dots, B_n\}$ be a stack composed of n blocks. If B_i rests on B_j , then B_j may apply an upward force of $f_{ij} \geq 0$ on B_i , in which case B_i will reciprocate by applying a downward force of the same magnitude on B_j . Since the blocks and table are frictionless, all the forces acting on them are vertical. The force f_{ij} may be assumed to be applied at a single point (x_{ij}, y_{ij}) in the contact interval I_{ij} . A downward gravitational force of unit magnitude is applied on B_i at its center of gravity $(x_i + \frac{1}{2}, y_i + \frac{h}{2})$.

DEFINITION 1. (EQUILIBRIUM) Let B be a homogeneous block of unit length and unit weight, and let a be the x -coordinate of its left edge. Let $(x_1, f_1), (x_2, f_2), \dots, (x_k, f_k)$ be the positions and the magnitudes of the upward forces applied to B along its bottom edge, and let $(x'_1, f'_1), (x'_2, f'_2), \dots, (x'_{k'}, f'_{k'})$ be the positions and magnitudes of the upward forces applied by B , along its top edge, on other blocks of the stack. Then, B is said to be in equilibrium under these collections of forces if and only if

$$\sum_{i=1}^k f_i = 1 + \sum_{i=1}^{k'} f'_i \quad , \quad \sum_{i=1}^k x_i f_i = (a + \frac{1}{2}) + \sum_{i=1}^{k'} x'_i f'_i .$$

The first equation says that the net force applied to B is zero while the second says that the net moment is zero.

DEFINITION 2. (BALANCE) A stack $\{B_1, \dots, B_n\}$ is said to be balanced if there exists a collection of forces acting between the blocks along their contact intervals, such that under this collection of forces, and the gravitational forces acting on them, all blocks are in equilibrium.

The stacks presented in Figures 1 are balanced. They are, however, *precariously* balanced, with some minute displacement of their blocks leading to imbalance and collapse. A stack can be said to be *stable* if all stacks obtained by sufficiently small displacements of its blocks are balanced. We do not make this definition formal as it is not used in the rest of the paper, though we refer to it in some informal discussions.

A schematic description of a stable stack and a collection of balancing forces acting between its blocks is given in Figure 5. Only upward forces are shown in the figure but corresponding downward forces are, of course, present. (We note in passing that balancing forces, when they exist, are in general not uniquely determined. This phenomenon is referred to as *static indeterminacy*.)

We usually adopt the convention that the blocks of a balanced stack are numbered consecutively from bottom to top and from left to right. Block B_1 is then the leftmost block in the lowest level while B_n is the rightmost block at the top level. For every $0 \leq i \leq n$, we let F_i be a collection of upward balancing forces applied by blocks in $\{B_0, B_1, \dots, B_i\}$ on blocks in $\{B_{i+1}, \dots, B_n\}$. (See Figure 5.) We refer to F_i as the collection of forces that cross the i -th *slice* of the stack.

Let us examine the relationship between two consecutive collections F_i and F_{i+1} . The only forces present in F_i but not in F_{i+1} are upward forces applied to B_i , while the only forces present in F_{i+1} but not in F_i are upward forces applied by B_i to blocks resting upon it. If we let $(x_1, f_1), (x_2, f_2), \dots, (x_k, f_k)$ be the positions and the magnitudes of the upward forces applied to B_i , and $(x'_1, f'_1), (x'_2, f'_2), \dots, (x'_{k'}, f'_{k'})$ be the positions and magnitudes of the upward forces applied by B_i , and if we let a be the x -coordinate of the left edge of B_i , we get by Definitions 1 and 2, that $\sum_{i=1}^k f_i = 1 + \sum_{i=1}^{k'} f'_i$ and $\sum_{i=1}^k x_i f_i = (a + \frac{1}{2}) + \sum_{i=1}^{k'} x'_i f'_i$. Block B_i thus *rearranges* the forces in the interval $[a, a + 1]$ in a way that preserves the total magnitude of the forces and their total moment, when its own weight is taken into account. Note that all forces of F_0 act in non-positive positions, and that if B_k is the most overhanging block in a stack and the overhang achieved by it is d , then the total magnitude of the forces in F_{k-1} that act at or beyond position $d-1$ should be at least 1. These simple observations play a central role in the rest of the paper.

2.3 The overhang problem. The natural formulation of the overhang problem is now:

What is the maximum overhang achieved by a balanced or stable n -block stack?

The main result of this paper is:

THEOREM 2.1. *The overhang achieved by a balanced n -block stack is at most $6n^{1/3}$.*

Our upper bound clearly applies also to the more demanding case of stable stacks, so, in combination with the above mentioned $\Omega(n^{1/3})$ construction from [PZ2006], we conclude:

COROLLARY 2.1. *The maximum overhang achievable by a balanced or stable n -block stack is $\Theta(n^{1/3})$.*

3 Mass movement problems

Our upper bound on the maximum achievable overhang is obtained by considering *mass movement* problems that are an abstraction of the way in which balancing forces “flow” through a stack of blocks. (See the discussion at the end of Section 2.2.)

In a mass movement problem we are required to transform an initial *mass distribution* into a mass distribution that satisfies certain conditions. The key condition is that a specified amount of mass be moved to or beyond a certain position. We can transform one mass distribution into another by performing local *moves* that redistribute mass within a given interval in a way that preserves the total mass and the center of mass. Our goal is then to show that many moves are required to accomplish the task. As can be seen, masses here correspond to forces, mass distributions correspond to collections of forces, and moves mimic the effects of blocks.

The mass movement problems considered are formally defined in Sections 3.1 and 3.2. The correspondence between the mass movement problems considered and the overhang problem is established in Section 3.3. The bounds on mass movement problems that imply Theorem 2.1 are then proved in Section 4.

3.1 Distributions.

DEFINITION 3. (DISTRIBUTIONS, SIGNED DISTRIBUTIONS) *A discrete mass distribution is a set $\mu = \{(x_1, m_1), (x_2, m_2), \dots, (x_k, m_k)\}$, where $k > 0$, x_1, x_2, \dots, x_k are real numbers, and $m_1, \dots, m_k > 0$. A signed distribution μ is defined the same way, but without the requirement that $m_1, m_2, \dots, m_k > 0$.*

If $\mu = \{(x_1, m_1), (x_2, m_2), \dots, (x_k, m_k)\}$ is a (signed) distribution, then for any set $A \subseteq \mathbb{R}$, we define

$$\mu(A) = \sum_{x_i \in A} m_i.$$

For brevity, we use $\mu(a)$ as a shorthand for $\mu(\{a\})$ and $\mu\{x > a\}$ as a shorthand for $\mu(\{x \mid x > a\})$. (Note that x here is a formal variable that does not represent a specific real number.) We similarly use $\mu\{x \geq a\}$, $\mu\{x < a\}$, $\mu\{a < x < b\}$, $\mu\{|x| \geq a\}$, etc., with the expected meanings.

We say that a (signed) distribution is on the interval $[a, b]$ if $\mu(x) = 0$ for every $x \notin [a, b]$.

For every $A \subseteq \mathbb{R}$, we let μ_A be the restriction of μ to A :

$$\mu_A = \{(x_i, m_i) \mid x_i \in A\}.$$

If μ_1 and μ_2 are two signed distributions, we let $\mu_1 + \mu_2$ and $\mu_1 - \mu_2$ be the signed distributions for which

$$\begin{aligned} (\mu_1 + \mu_2)(x) &= \mu_1(x) + \mu_2(x), & \text{for every } x \in \mathbb{R}, \\ (\mu_1 - \mu_2)(x) &= \mu_1(x) - \mu_2(x), & \text{for every } x \in \mathbb{R}. \end{aligned}$$

DEFINITION 4. (MOMENTS) Let $\mu = \{(x_1, m_1), (x_2, m_2), \dots, (x_k, m_k)\}$ be a signed distribution and let $j \geq 0$ be an integer. The j -th moment of μ is defined to be:

$$M_j[\mu] = \sum_{i=1}^k m_i x_i^j.$$

Note that $M_0[\mu]$ is the total mass of μ , $M_1[\mu]$ is the torque of μ , with respect to the origin, and $M_2[\mu]$ is the moment of inertia of μ , again with respect to the origin. If $M_0[\mu] \neq 0$, we denote $M_1[\mu]/M_0[\mu]$, the center of mass of μ , by $C[\mu]$.

Less standard, but crucial for our analysis, is the following definition.

DEFINITION 5. (SPREAD) The spread of a distribution $\mu = \{(x_1, m_1), (x_2, m_2), \dots, (x_k, m_k)\}$ is defined as follows:

$$S[\mu] = \sum_{i < j} |x_i - x_j| m_i m_j.$$

If $M_0[\mu] = 1$, then μ defines a discrete random variable X for which $\Pr[X = x] = \mu(x)$, for every $x \in \mathbb{R}$. The spread $S[\mu]$ is then half the average distance between two independent drawings from μ . We also then have $M_1[\mu] = E[X]$ and $M_2[\mu] = E[X^2]$. If $M_1[\mu] = E[X] = 0$, then $M_2[\mu] = E[X^2] = \text{Var}[X]$. It is also worthwhile noting that if μ_1 and μ_2 are two distributions then, for any $k \geq 0$, $M_k[\mu_1 + \mu_2] = M_k[\mu_1] + M_k[\mu_2]$, i.e., M_k is a linear operator.

An inequality that proves very useful in the sequel is the following:

LEMMA 3.1. For any discrete distribution μ we have $S[\mu]^2 \leq \frac{1}{3} M_2[\mu] M_0[\mu]^3$.

The lemma is essentially the case $n = 2$ of a more general result proved by Plackett [P1947]. For details, the reader is referred to the full version of this extended abstract [PPTWZ2007].

3.2 Mass redistribution moves.

DEFINITION 6. (MOVES) A move $v = ([a, b], \delta)$ consists of an interval $[a, b]$ and a signed distribution δ on $[a, b]$ with $M_0[\delta] = M_1[\delta] = 0$. A move v can be applied to a distribution μ if the signed distribution $\mu' = \mu + \delta$ is a distribution, in which case we denote the result μ' of this application by $v\mu$. We refer to $\frac{a+b}{2}$ as the center of the move. Unless otherwise stated, the moves we consider operate on intervals of length 1, i.e., $b - a = 1$.

Note that if v is a move and $\mu' = v\mu$, then $M_0[\mu'] = M_0[\mu]$, $M_1[\mu'] = M_1[\mu]$ and consequently $C[\mu'] = C[\mu]$.

A sequence $V = \langle v_1, v_2, \dots, v_\ell \rangle$ of moves and an initial distribution μ_0 naturally define a sequence of distributions $\mu_0, \mu_1, \dots, \mu_\ell$, where $\mu_i = v_i \mu_{i-1}$ for $1 \leq i \leq \ell$. (It is assumed here that v_i can indeed be applied to μ_{i-1} .) We let $V\mu_0 = \mu_\ell$.

Moves and sequences of moves simulate the behavior of *weightless* blocks and stacks. However, the blocks that we are interested in have unit weight. Instead of explicitly taking into account the weight of the blocks, as we briefly do in Section 3.3, it turns out that it is enough for our purposes to impose a natural restriction on the move sequences considered. We start with the following definition:

DEFINITION 7. (μ_{\max}) If $\mu_0, \mu_1, \dots, \mu_\ell$ is a sequence of distributions, and $a \in \mathbb{R}$, we define

$$\mu_{\max}\{x > a\} = \max_{0 \leq i \leq \ell} \mu_i\{x > a\}.$$

Expressions such as $\mu_{\max}\{x \geq a\}$, $\mu_{\max}\{x < a\}$ and $\mu_{\max}\{x \leq a\}$ are defined similarly.

DEFINITION 8. (WEIGHT-CONSTRAINED SEQUENCES)

A sequence $V = \langle v_1, v_2, \dots, v_\ell \rangle$ of moves that generates a sequence $\mu_0, \mu_1, \dots, \mu_\ell$ of distributions is said to be weight-constrained, with respect to μ_0 if, for every $a \in \mathbb{R}$, the number of moves in V centered in (a, ∞) is at most $\mu_{\max}\{x > a\}$.

The two main technical results of this paper are the following theorems.

THEOREM 3.1. If a distribution ν is obtained from a distribution μ with $\mu\{x \leq 0\} \leq n$ and $\mu\{x > 0\} = 0$, where $n \geq 1$, by a weight-constrained move sequence, then $\nu\{x \geq 6n^{1/3} - 1\} = 0$.

For general move sequences we have the following almost tight result, which might be of some independent interest. In particular, it shows that the weight constraint only has a logarithmic effect on the maximal overhang.

THEOREM 3.2. If a distribution ν is obtained from a distribution μ with $\mu\{x \leq 0\} \leq n$ and $\mu\{x > 0\} = 0$, where $n \geq 1$, by a move sequence of length at most n , then $\nu\{x \geq 2n^{1/3} \log_2 n\} < 1$.

We show next that Theorem 3.1 does indeed imply Theorem 2.1, the main result of this extended abstract.

3.3 From overhang to mass movement. The moves of Definition 6 capture the essential effect that a block can have on the collections of forces within a stack. They fail to take into account, however, the fact that the *weight* of a block is “used up” by the move and is

then lost. To faithfully simulate the effect of unit weight blocks we introduce the slightly modified definition of *lossy moves*:

DEFINITION 9. (LOSSY MOVES) *If $v = ([a, b], \delta)$ is a move, then the lossy move v^\downarrow associated with it is $v^\downarrow = ([a, b], \delta^\downarrow)$, where $\delta^\downarrow = \delta - \{(\frac{a+b}{2}, 1)\}$. A lossy move v^\downarrow can be applied to a distribution μ if $\mu' = \mu + \delta^\downarrow$ is a distribution, in which case we denote the result μ' of this application by $v^\downarrow \mu$.*

Note that if $v^\downarrow = ([a, b], \delta^\downarrow)$ is a lossy move and $\mu' = v^\downarrow \mu$, then $M_0[\mu'] = M_0[\mu] - 1$ and $M_1[\mu'] = M_1[\mu] - \frac{a+b}{2}$. Hence, lossy moves do not preserve total mass or center of mass.

If $V = \langle v_1, v_2, \dots, v_\ell \rangle$ is a sequence of moves, we let $V^\downarrow = \langle v_1^\downarrow, v_2^\downarrow, \dots, v_\ell^\downarrow \rangle$ be the corresponding sequence of lossy moves. If μ_0 is an initial distribution, we can naturally define the sequence of distributions $\mu_0, \mu_1, \dots, \mu_\ell$, where $\mu_i = v_i^\downarrow \mu_{i-1}$ for $1 \leq i \leq \ell$, obtained by applying V^\downarrow to μ_0 .

A collection of forces F_i may also be viewed as mass distribution. The following lemma is now a simple formulation of the definitions and the discussion of Section 2.2:

LEMMA 3.2. *Let $\{B_1, B_2, \dots, B_n\}$ be a balanced stack. Let F_i be a collection of balancing forces acting between $\{B_0, \dots, B_i\}$ and $\{B_{i+1}, \dots, B_n\}$, for $0 \leq i \leq n$. Let x_i be the x -coordinate of the left edge of B_i . Then, F_{i+1} can be obtained from F_i by a lossy move in the interval $[x_i, x_i+1]$.*

As an immediate corollary, we get:

LEMMA 3.3. *If there is a stack composed of n blocks of length 1 and weight 1 that achieves an overhang of d , then there is sequence of at most $n-1$ lossy moves that transforms a distribution μ with $M_0[\mu] = \mu\{x \leq 0\} = n$ and $\mu\{x > 0\} = 0$ into a distribution μ' with $\mu'\{x \geq d-1\} \geq 1$.*

Proof. Let $\{B_1, B_2, \dots, B_n\}$ be a balanced stack and let B_k be a block in it that achieves an overhang of d . As before, we let F_i be a collection of balancing forces acting between $\{B_0, \dots, B_i\}$ and $\{B_{i+1}, \dots, B_n\}$. We let $\mu = F_0$ and $\mu' = F_{k-1}$. It follows from Lemma 3.2 that μ' may be obtained from μ by a sequence of $k-1$ lossy moves. As all the forces in $\mu = F_0$ are forces applied by the table B_0 , and as the table supports the weight of the n blocks of the stack, we have $M_0[\mu_0] = \mu_0\{x \leq 0\} = n$ and $\mu\{x > 0\} = 0$. As the forces in $\mu' = F_{k-1}$ must at least support the weight of B_k , we have $\mu'\{d-1 \leq x \leq d\} \geq 1$.

The next simple lemma shows that sequences of lossy moves can be easily converted into weight-constrained sequences of moves and distributions that “dominate” the original sequence.

LEMMA 3.4. *If $\mu_0, \mu_1, \dots, \mu_\ell$ is a sequence of distributions obtained by a sequence of lossy moves, then there exists a sequence of distributions $\mu'_0, \mu'_1, \dots, \mu'_\ell$ obtained by a weight-constrained sequence of moves such that $\mu'_0 = \mu_0$, and $\mu'_i(x) \geq \mu_i(x)$, for every $1 \leq i \leq \ell$ and $x \in \mathbb{R}$.*

Proof. The sequence $\mu'_0, \mu'_1, \dots, \mu'_\ell$ is obtained by performing exactly the same moves used to obtain the sequence $\mu_0, \mu_1, \dots, \mu_\ell$, treating them now as moves rather than lossy moves. More formally, if $\mu_i = v_i^\downarrow \mu_{i-1}$, we let $\mu'_i = v_i \mu'_{i-1}$. If $v_i = ([a - \frac{1}{2}, a + \frac{1}{2}], \delta)$, then μ'_i now has an extra mass of size 1 at a . This mass is *frozen*, and will not be touched by subsequent moves. Hence, if k moves have their center beyond position a , then $\mu'_{\max}\{x > a\} \geq \mu'_\ell\{x > a\} \geq k$, as required by the definition of weight-constrained sequences.

It is now easy to see that Theorem 3.1 together with Lemmas 3.2 and 3.3 imply Theorem 2.1.

4 Bounds on mass movement problems

This section is devoted to the proofs of Theorems 3.1 and 3.2. As mentioned, Theorem 3.1 implies Theorem 2.1, which states that an n -block stack can have an overhang of at most $6n^{1/3}$.

We begin by considering an important class of moves:

DEFINITION 10. (EXTREME MOVES) *An extreme move \bar{v} is defined solely as an interval $[a, b]$. An extreme move \bar{v} can be applied to any distribution μ resulting in the distribution $\mu' = \bar{v}\mu$ such that $\mu'\{a < x < b\} = 0$, $\mu'(x) = \mu(x)$ for every $x \notin [a, b]$, $M_0[\mu] = M_0[\mu']$ and $M_1[\mu] = M_1[\mu']$. In other words, an extreme move moves all the mass in the interval $[a, b]$ into the endpoints of this interval while maintaining the center of mass. If v is a move on an interval $[a, b]$, we let \bar{v} denote the extreme move on $[a, b]$. If V is a sequence of moves, we let \bar{V} denote the corresponding sequence of extreme moves.*

Closely related to Lemma 3.1 is the following lemma :

LEMMA 4.1. *If μ_1 is obtained from μ_0 by an extreme move (in an interval of length 1) then*

$$S[\mu_1] - S[\mu_0] \geq 3(M_2[\mu_1] - M_2[\mu_0])^2.$$

The proof of Lemma 4.1 is based on the Cauchy-Schwartz inequality, and is deferred to the full version of this paper [PPTWZ2007]. Another important lemma whose proof is also deferred is that if we make all moves extreme, the second moment increases:

LEMMA 4.2. *If V is a sequence of moves that can be applied to μ , then $M_2[V\mu] \leq M_2[\bar{V}\mu]$.*

4.1 Spread vs. second moment. We now obtain our first bound for mass movement problems. The bound relies heavily on Lemma 3.1 that relates the spread and second moment of a distribution, on Lemma 4.1 that relates differences in spread to differences in second moments, and finally, on Lemma 4.2 that states that converting moves to extreme moves can only increase the second moment.

LEMMA 4.3. *Any sequence of moves that transforms the distribution $\mu = \{(0, 1)\}$ into a distribution ν with $\nu\{|x| \geq d\} \geq p$, where $d > 0$ and $0 < p < 1$, must contain at least $(3p)^{3/2}d^3$ moves.*

Proof. Let $\mu_0, \mu_1, \dots, \mu_\ell$ be the sequence of distributions obtained by applying a sequence V of ℓ moves to $\mu_0 = \{(0, 1)\}$, and suppose that $\mu_\ell\{|x| \geq d\} \geq p$. By the definition of the second moment we have $M_2[\mu_\ell] \geq pd^2$.

Let $\bar{\mu}_0, \bar{\mu}_1, \dots, \bar{\mu}_\ell$ be the sequence of distributions obtained by applying the sequence \bar{V} of the extreme moves corresponding to the moves of V on $\bar{\mu}_0 = \mu_0 = \{(0, 1)\}$. By Lemma 4.2, we get that

$$M_2[\bar{\mu}_\ell] \geq M_2[\mu_\ell] \geq pd^2.$$

By Lemma 3.1 we have

$$\begin{aligned} \frac{M_2[\bar{\mu}_\ell]^2}{S[\bar{\mu}_\ell]} &= \left(\frac{M_2[\bar{\mu}_\ell]M_0[\bar{\mu}_\ell]^3}{S[\bar{\mu}_\ell]^2} \right)^{1/2} M_2[\bar{\mu}_\ell]^{3/2} \\ &\geq \sqrt{3} M_2[\bar{\mu}_\ell]^{3/2} \geq \sqrt{3} p^{3/2} d^3. \end{aligned}$$

Let $h_i = M_2[\bar{\mu}_i] - M_2[\bar{\mu}_{i-1}]$, for $1 \leq i \leq \ell$. As $M_2[\bar{\mu}_0] = 0$, we clearly have

$$M_2[\bar{\mu}_\ell] = \sum_{i=1}^{\ell} h_i.$$

By Lemma 4.1, we get that

$$S[\bar{\mu}_\ell] \geq 3 \sum_{i=1}^{\ell} h_i^2.$$

Using the Cauchy-Schwartz inequality to justify the second inequality below, we get:

$$S[\bar{\mu}_\ell] \geq 3 \sum_{i=1}^{\ell} h_i^2 \geq 3 \frac{(\sum_{i=1}^{\ell} h_i)^2}{\ell} = 3 \frac{M_2[\bar{\mu}_\ell]^2}{\ell}.$$

Thus, as claimed,

$$\ell \geq 3 \frac{M_2[\bar{\mu}_\ell]^2}{S[\bar{\mu}_\ell]} \geq (3p)^{3/2} d^3.$$

4.2 Mirroring. The main result of this section is:

THEOREM 4.1. *Let $\mu_0, \mu_1, \dots, \mu_\ell$ be a sequence of distributions obtained by applying a sequence of moves to an initial distribution μ_0 with $\mu_0\{x > r\} = 0$. If $\mu_{\max}\{x > r\} \leq m$ and $\mu_{\max}\{x \geq r + d\} \geq pm$, where $d > 1$ and $0 < p < 1$, then the sequence of moves must contain at least $\sqrt{3}p^{3/2}(d - \frac{1}{2})^3$ moves whose centers are in $(r + \frac{1}{2}, \infty)$.*

The theorem follows immediately from the following lemma by shifting coordinates and renormalizing masses.

LEMMA 4.4. *Let $\mu_0, \mu_1, \dots, \mu_\ell$ be a sequence of distributions obtained by applying a sequence of moves to an initial distribution μ_0 with $\mu_0\{x > -\frac{1}{2}\} = 0$. If $\mu_{\max}\{x > -\frac{1}{2}\} \leq 1$ and $\mu_{\max}\{x \geq d\} \geq p$, where $d > \frac{1}{2}$ and $0 < p < 1$, then the sequence of moves must contain at least $\sqrt{3}p^{3/2}d^3$ moves whose centers are at strictly positive positions.*

Proof. We may assume, without loss of generality, that the first move in the sequence moves some mass from $(-\infty, \frac{1}{2})$ into $(\frac{1}{2}, \infty)$ and that the last move moves some mass from $(-\infty, d)$ to $[d, \infty)$. Hence, the center of the first move must be in $(-1, 0]$ and the center of the last move must be at a positive position.

We shall show how to transform the sequence of distributions $\mu_0, \mu_1, \dots, \mu_\ell$ into a sequence of distributions $\mu'_0, \mu'_1, \dots, \mu'_{\ell'}$ obtained by applying a sequence of ℓ' moves, such that $\mu'_0 = \{(0, 1)\}$, $\mu'_{\ell'}\{|x| \geq d\} \geq p$, and such that the number of moves ℓ' in the new sequence is at most three times the number ℓ^+ of positively centered move in the original sequence. The claim of the lemma would then follow immediately from Lemma 4.3.

The first transformation is “negative truncation”, where in each distribution μ_i , we shift mass from the interval $(-\infty, -\frac{1}{2})$ to the point $-\frac{1}{2}$. Formally the resulting distribution $\bar{\mu}_i$ is defined by

$$\bar{\mu}_i(x) = \begin{cases} \mu_i(x) & \text{if } x > -\frac{1}{2} \\ 1 - \mu_i\{x > -\frac{1}{2}\} & \text{if } x = -\frac{1}{2} \\ 0 & \text{if } x < -\frac{1}{2} \end{cases}.$$

Note that the total mass of each distribution is 1 and that $\bar{\mu}_0 = \{(-\frac{1}{2}, 1)\}$. Let $\delta_i = \mu_i - \mu_{i-1}$ be the signed distribution associated with the move that transforms μ_{i-1} into μ_i and let $[c_i - \frac{1}{2}, c_i + \frac{1}{2}]$ be the interval in which it operates. For brevity, we refer to δ_i as the move itself, with its center c_i clear from the context. We now compare the transformed “moves” $\bar{\delta}_i = \bar{\mu}_i - \bar{\mu}_{i-1}$ with the original moves $\delta_i = \mu_i - \mu_{i-1}$. If $c_i > 0$, then δ_i acts above $-\frac{1}{2}$ and $\bar{\delta}_i = \delta_i$. If $c_i \leq -1$, then δ_i acts at or below $-\frac{1}{2}$, so $\bar{\delta}_i$ is null and $\bar{\mu}_i = \bar{\mu}_{i-1}$. In the transformed sequence, we skip all such null moves. The

remaining case is when the center c_i of δ_i is in $(-1, 0]$. In this case $\vec{\delta}_i$ acts within $[-\frac{1}{2}, \frac{1}{2}]$, and we view it as centered at 0. However, typically $\vec{\delta}_i$ does not define a valid move as it may change the center of gravity. We call such $\vec{\delta}_i$ *semi-moves*. If we have two consecutive semi-moves $\vec{\delta}_i$ and $\vec{\delta}_{i+1}$, we combine them into a single semi-move $\vec{\delta}_i + \vec{\delta}_{i+1}$, taking $\vec{\mu}_{i-1}$ directly to $\vec{\mu}_{i+1}$. In the resulting negatively truncated and simplified sequence, we know that at least every alternate move is an original, positively centered, move. Since the last move in the original sequence was positively centered we conclude:

CLAIM 1. *The sequence obtained by the negative truncation transformation and the subsequent clean-up is composed of original positively centered moves and semi-moves (acting within $[-\frac{1}{2}, \frac{1}{2}]$). The sequence begins with a semi-move and at most half of its elements are semi-moves.*

Next, we create a reflected copy of the negatively truncated distributions. The reflected copy $\bar{\mu}_i$ of $\vec{\mu}_i$ is defined by

$$\bar{\mu}_i(x) = \vec{\mu}_i(-x), \quad \text{for every } x \in \mathbb{R}.$$

We similarly define the reflected (semi-)moves $\bar{\delta}_i = \vec{\mu}_i - \bar{\mu}_{i-1}$. We can now define the mirrored distributions

$$\bar{\vec{\mu}}_i = \vec{\mu}_i + \bar{\mu}_i.$$

Note that $\bar{\vec{\mu}}_0 = \vec{\mu}_0 + \bar{\mu}_0 = \{(-\frac{1}{2}, 1), (\frac{1}{2}, 1)\}$. The distribution $\bar{\vec{\mu}}_i$ may be obtained from $\bar{\vec{\mu}}_{i-1}$ by applying the (semi-)move $\bar{\delta}_i$, resulting in the distribution $\bar{\vec{\mu}}_i + \bar{\mu}_{i-1}$, and then the (semi-)move $\vec{\delta}_i$, resulting in $\bar{\vec{\mu}}_i + \bar{\mu}_i = \bar{\vec{\mu}}_i$. The $\bar{\vec{\mu}}_i$ sequence is therefore obtained by interleaving the (semi-)moves $\bar{\delta}_i$ with their reflections $\vec{\delta}_i$. Now comes a key observation.

CLAIM 2. *If $\vec{\delta}_i$ and $\bar{\delta}_i$ are semi-moves, then their sum $\bar{\vec{\delta}}_i = \vec{\delta}_i + \bar{\delta}_i$ defines an ordinary move centered at 0 and acting on $[-\frac{1}{2}, \frac{1}{2}]$.*

Proof. Both $\vec{\delta}_i$ and $\bar{\delta}_i$ preserve the total mass. As $\vec{\delta}_i$ is symmetric about 0, it cannot change the center of mass.

As suggested by the above observation, if $\vec{\delta}_i$ and $\bar{\delta}_i$ are semi-moves, we combine them into a single ordinary move $\bar{\vec{\delta}}_i$ centered at 0. We thus obtain a sequence of at most $3\ell^+$ moves, where ℓ^+ is the number of positively centered moves in the original sequence, that transforms $\vec{\mu}_0$ to $\bar{\vec{\mu}}_\ell$.

Recall from Claim 1 that the first “move” $\vec{\delta}_1$ in the negatively truncated sequence is a semi-move. The first move $\bar{\vec{\delta}}_1$, obtained by combining $\vec{\delta}_1$ and $\bar{\delta}_1$, is therefore a move acting on $[-\frac{1}{2}, \frac{1}{2}]$. We now replace the initial distribution $\vec{\mu}_0 = \{(-\frac{1}{2}, 1), (\frac{1}{2}, 1)\}$ by the

distribution $\mu'_0 = \{(0, 2)\}$, which has the same center of gravity, and replace the first move by $\delta'_1 = \vec{\delta}_1 + \{(-\frac{1}{2}, 1), (0, -2), (\frac{1}{2}, 1)\}$. The distribution after the first move is then again $\vec{\mu}_1$.

We have thus obtained a sequence of at most $3\ell^+$ moves that transforms $\mu'_0 = \{(0, 2)\}$ into a distribution $\nu' = \vec{\mu}_\ell$ with $\nu'\{|x| \geq d\} \geq 2p$. Scaling these distributions and moves by a factor of 2, we get, by Lemma 4.3, that $3\ell^+ \geq (3p)^{3/2}d^3$, as claimed.

4.3 Proofs of Theorems 3.1 and 3.2. We prove the following theorem which easily implies Theorem 3.1.

THEOREM 4.2. *Let $\mu_0, \mu_1, \dots, \mu_\ell$ be a sequence of distributions obtained by applying a constrained sequence of moves to an initial distribution μ_0 with $\mu_0\{x > r\} = 0$. If $\mu_{\max}\{x > r\} \leq n$, where $n \geq \frac{1}{5}$, then $\mu_{\max}\{x > r + 6n^{1/3} - 1\} = 0$.*

Proof. The proof is by induction on n . If $n < 1$ then there is no move with center greater than r and hence $\mu_{\max}\{x > r + \frac{1}{2}\} = 0$. Since $6(\frac{1}{5})^{1/3} - 1 > 1/2$, the result clearly holds.

Suppose now that $\mu_{\max}\{x > r\} = n$ and that the result holds for all $\frac{1}{5} \leq n' < n$. Let u be the largest number for which $\mu_{\max}\{x \geq r + u\} > \frac{n}{5}$. As the distributions μ_i are discrete, it follows that $\mu_{\max}\{x > r + u\} \leq \frac{n}{5}$. As $u \geq 0$, we have $\mu_0\{x > r + u\} = 0$. By the induction hypothesis with r replaced by $r + u$, we therefore get that

$$\mu_{\max}\left\{x > r + u + 6\left(\frac{n}{5}\right)^{1/3} - 1\right\} = 0.$$

As $\mu_0\{x > r\} = 0$, $\mu_{\max}\{x > r\} \leq n$ and $\mu_{\max}\{x \geq r + u\} > \frac{n}{5}$, we get by Theorem 4.1 that the sequence must contain at least $\sqrt{3}(\frac{1}{5})^{3/2}(u - \frac{1}{2})^3 > \frac{1}{7}(u - \frac{1}{2})^3$ moves whose centers are positive. As the sequence of moves is constrained, and as $\mu_{\max}\{x > r\} \leq n$, there can be at most n such moves with centers greater than r , i.e., $\frac{1}{7}(u - \frac{1}{2})^3 \leq n$. Hence $u \leq (7n)^{1/3} + \frac{1}{2}$, and so

$$\begin{aligned} u + 6\left(\frac{n}{5}\right)^{1/3} - 1 &\leq \left(7^{1/3} + 6 \cdot \left(\frac{1}{5}\right)^{1/3}\right)n^{1/3} - \frac{1}{2} \\ &< 5.5n^{1/3} - \frac{1}{2} \leq 6n^{1/3} - 1, \end{aligned}$$

for $n \geq 1$. This proves the induction step and completes the proof.

Modulo the details deferred to the full version of this extended abstract [PPTWZ2007], this completes the proof of our main result that the maximum overhang that can be achieved using n blocks is at most $6n^{1/3}$.

Next, we prove the following theorem which easily implies Theorem 3.2.

THEOREM 4.3. *Let $\mu_0, \mu_1, \dots, \mu_n$ be a sequence of distributions obtained by applying a sequence of n moves to an initial distribution μ_0 with $\mu_0\{x \leq 0\} \leq n$ and $\mu_0\{x > 0\} = 0$, where $n \geq 2$. Then $\mu_n\{x > 2n^{1/3} \log_2 n\} < 1$.*

Proof. Suppose that $2^k \leq n < 2^{k+1}$, where $k \geq 0$. For $1 \leq i \leq k$, let u_i be the largest number for which $\mu_{\max}\{x \geq u_i\} \geq \frac{n}{2^i}$. By the discreteness of the distributions we again have $\mu_{\max}\{x > u_i\} < \frac{n}{2^i}$. Let $u_0 = 0$. Assume, for the sake of contradiction, that $\mu_n\{x > 2n^{1/3} \log_2 n\} \geq 1$. Then, $u_k \geq 2kn^{1/3}$. There is then at least one value of i for which $u_i - u_{i-1} \geq 2n^{1/3}$. By Theorem 4.1, applied with $r = u_{i-1}$ and $d = u_i - u_{i-1}$, we conclude that the sequence must contain more than n moves, a contradiction.

We believe that a stronger version of the theorem, which states under the same conditions that $\mu_n\{x > cn^{1/3}(\log_2 n)^{2/3}\} < 1$, for some $c > 0$, actually holds. This would match an example supplied by Johan Håstad. Theorem 4.3 (and Theorem 3.2) imply an almost tight bound on an interesting variant of the overhang problem that involves weightless blocks, as discussed in Section 5.

5 Concluding remarks and open problems

We have shown that the maximum overhang achieved using n homogeneous, frictionless blocks of unit length is at most $6n^{1/3}$. Thus, the constructions of [PZ2006] cannot be improved by more than a constant factor, establishing order $n^{1/3}$ as the asymptotic answer to the age-old overhang problem.

The discussions and results presented so far all refer to the standard two-dimensional version of the overhang problem. Our results hold, however, in greater generality. We briefly discuss some natural generalizations and variants of the overhang problem for which our bounds still apply.

In Section 2 we stipulated that all blocks have a given height h . It is easy to see, however, that all our results remain valid even if blocks have different heights, but still have unit length and unit weight. In particular, blocks are allowed to degenerate into *sticks*, i.e., have height 0. Also, even though we required blocks not to overlap, we did not use this condition in any of our proofs.

Loaded stacks, introduced in [PZ2006], are stacks composed of standard unit length and unit weight blocks, and *point weights* that can have arbitrary weight. (Point weights may be considered to be blocks of zero height and length, but nonzero weight.) Our results, with essentially no change, imply that loaded stacks of total weight n can have an overhang of at most $6n^{1/3}$.

What happens when we are allowed to use blocks of different lengths and weights? Our results can be generalized in a fairly straightforward way to show that if a block of length ℓ has weight proportional to ℓ^3 , as would be the

case if all blocks were similar three-dimensional cuboids, then the overhang of a stack of total weight n is again of order at most $n^{1/3}$. It is amusing to note that in this case an overhang of order $n^{1/3}$ can be obtained by stacking n unit-length blocks as in the construction of [PZ2006], or simply by balancing a single block of length $n^{1/3}$ and weight n at the edge of the table! Might this mean that there is some physical principle that could have told us, without all the calculations, that the right answer to the original overhang problem has to be of order $n^{1/3}$?

Theorem 3.2 supplies an almost tight upper bound for the following variant of the overhang problem: how far away from the edge of a table can a mass of weight 1 be supported using n *weightless* blocks of length 1 and a collection of point weights of total weight n ? The overhang in this case beats the classical one by a factor of between $\log^{2/3} n$ and $\log n$.

In all variants considered so far, blocks were assumed to have their longest edges parallel to the table's surface and perpendicular to its edge. The assumption of no friction then immediately implied that all forces within a stack are vertical, and our analysis, which assumes that there are no horizontal forces, was applicable. A nice argument, communicated to us by Harry Paterson, shows that in the frictionless two-dimensional case, horizontal forces cannot be present even if some of the blocks are tilted. Our results thus apply also in this case.

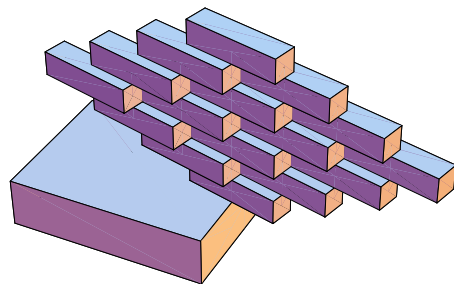


Figure 6: A “skintled” 4-diamond.

We believe that our bounds apply, with slightly adjusted constants, also in three dimensions, but proving so remains an open problem. Overhang larger by a factor of $\sqrt{1+w^2}$ may be obtained with $1 \times w \times h$ blocks, where $h \leq w \leq 1$, using a technique called *skintling* (see Figure 6). In skintling (a term we learned from an edifying conversation with John H. Conway about brick-laying) each block is rotated about its vertical axis, so that—in our case—the diagonal of its bottom face is perpendicular to the edge of the table. With suitably adjusted notion of length, however, our bounds apply to any three-dimensional construction that can be balanced using vertical forces. It is an interesting open problem whether there exist three-dimensional stacks composed of frictionless, possibly tilted, blocks that can only be balanced with the aid of some nonvertical forces. (We know that this is

possible if the blocks are *nonhomogeneous* and are of different sizes.) As mentioned, we believe that our bounds do apply in three dimensions even if it turns out that nonvertical forces are sometimes useful, but proving this requires some additional arguments.

We end by commenting on the tightness of the analysis presented in this paper. Our main result is a $6n^{1/3}$ upper bound on the overhang that may be obtained using n blocks. As we indicate in the full paper, this bound can be easily improved to about $4.5n^{1/3}$ for sufficiently large values of n . Various other small improvements in the constants are possible. For example, a careful examination of our proofs reveals that whenever we apply Lemma 4.1, the distribution μ_0 contains at most three masses in the interval acted upon by the move that produces μ_1 . (This follows from the fact that a block can rest upon at most three other blocks.) The constant 3 appearing in Lemma 4.1 can then be improved, though it is optimal when no assumption regarding the distribution μ_0 is made. We believe, however, that new ideas would be needed to reduce the upper bound to below, say, $3n^{1/3}$.

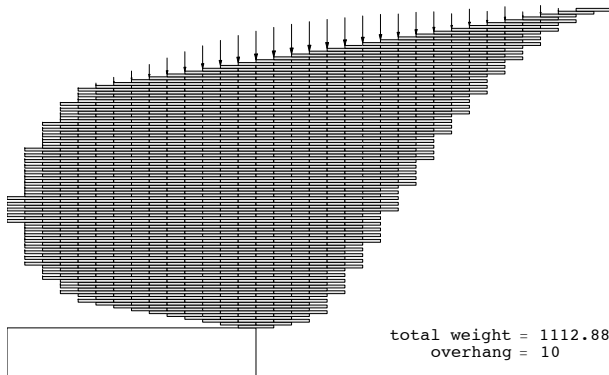


Figure 7: An “oil-lamp”-shaped stack

As mentioned, Paterson and Zwick [PZ2006] describe simple balanced n -block stacks that achieve an overhang of about $0.57n^{1/3}$. They also present some numerical evidence that suggests that the overhang that can be achieved using n blocks is at least $1.02n^{1/3}$, for large values of n . These larger overhangs are obtained using stacks that are shaped like the “oil-lamp” depicted in Figure 7. For more details on the figure and on “oil-lamp” constructions, see [PZ2006]. (The stack shown in the figure is actually a loaded stack, as defined above, with the external forces shown representing the point weights.)

A small gap still remains between the best upper and lower bounds currently available for the overhang problem, though they are both of order $n^{1/3}$. Determining a constant c such that the maximum overhang achievable using n blocks is asymptotically $cn^{1/3}$ is a challenging open problem.

Acknowledgements

We would like to thank John H. Conway, Johan Håstad, Harry Paterson, Anders Thorup and Benjy Weiss for useful discussions observations, some of which appear, with due credit, in the full paper.

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