

3 The Dot-town Suicides

Each resident of Dot-town carries a red or blue dot on his (or her) forehead, but if he ever figures out what color it is he kills himself. Each day the residents gather; one day a stranger comes and tells them something—*anything*—non-trivial about the number of blue dots. Prove that eventually every resident kills himself.

Comment: “Non-trivial” means here that there is some number of blue dots for which the statement would not have been true. Thus we have a frighteningly general version of classical problems involving knowledge about knowledge.

4 Unwanted Expansion

Suppose you have an algebraic expression involving variables, addition, multiplication, and parentheses. You repeatedly attempt to expand it using the distributive law. How do you know that the expression doesn’t continue to expand forever?

Comment: Note that applying the distributive law to, say, the outer product in $(x + y)(s(u + v) + t)$, yields $x(s(u + v) + t) + y(s(u + v) + t)$ which has more parentheses than before.

5 Love in Kleptopia

Jan and Maria have fallen in love (via the internet) and Jan wishes to mail her a ring. Unfortunately, they live in the country of Kleptopia where anything sent through the mail will be stolen unless it is enclosed in a padlocked box. Jan and Maria each have plenty of padlocks, but none to which the other has a key. How can Jan get the ring safely into Maria’s hands?

6 Winning at Wimbledon

As a result of temporary magical powers, you have made it to the Wimbledon finals and are playing Roger Federer for all the marbles. However, your powers cannot last the whole match. What score do you want it to be when they disappear, to maximize your chances of hanging on for a win?

7 The Random Native

The logician of Martin Gardner’s first *Scientific American* collection is again visiting the South Seas, and is as before at a fork, wanting to know which of two roads leads to the village. This time, present are three willing natives, one each from a tribe of invariable truth-tellers, a tribe of invariable liars, and a tribe of random answerers. Of course the logician doesn’t know which native is from which tribe. Moreover, he is permitted to ask only two yes-or-no questions, each question being directed to just one native. Can he get the information he needs?

8 solution to Names in Boxes

This puzzle has a short but fascinating history. Devised by Danish computer scientist Peter Bro Miltersen, a version appeared in a prize-winning paper of his and Anna Gal's.² But Miltersen didn't think there was a solution until one was pointed out to him over lunch by colleague Sven Skyum. Eventually, the puzzle reached me (in a slightly more complex form than presented here) via Dorit Aharonov.

To solve it, the prisoners must first agree on a random labeling of the boxes by their own names. (The point of making it random is that it makes it impossible for the warden to place names in boxes in such a way as to foil the protocol described next.) When admitted to the room, each prisoner inspects his own box (that is, the box with which his own name has been associated). He then looks into the box belonging to the name he just found, and then into the box belonging to the name he found in the second box, etc. until he either finds his own name, or has opened 50 boxes.

That's the strategy; now, why on earth should it work? Well, the process which assigns to a box's owner the name found in his box is a permutation of the 100 names, chosen uniformly at random from the set of all such permutations. Each prisoner is following a cycle of the permutation, beginning with his box and (if he doesn't run over the 50-box limit) ending with his name on a piece of paper. If it happens that the permutation *has no cycles of length greater than 50*, this process will work every time and the prisoners will be spared.

In fact, the probability that a uniformly random permutation of the numbers from 1 to $2n$ contains no cycle of length greater than n is at least $1 - \ln 2$ —about 30.6853%.

To see this, let $k > n$ and count the permutations having a cycle C of length exactly k . There are $\binom{2n}{k}$ ways to pick the entries in C , $(k-1)!$ ways to order them, and $(2n-k)!$ ways to permute the rest; the product of these numbers is $(2n)!/k$. Since at most one k -cycle can exist in a given permutation, the probability that there is one is exactly $1/k$.

It follows that the probability that there is no long cycle is

$$1 - \frac{1}{n+1} - \frac{1}{n+2} - \dots - \frac{1}{2n} = 1 - H_{2n} + H_n$$

where H_m is the sum of the reciprocals of the first m positive integers, approximately $\ln m$. Thus our probability is about $1 - \ln 2n + \ln n = 1 - \ln 2$, and in fact is always a bit larger. For $n = 50$ we get that the prisoners survive with probability 31.1827821%.

Eugene Curtin and Max Washauer have recently proved that this solution cannot be improved upon.

9 solution to Boxes in Boxes

This lovely puzzle was communicated to me by Anthony Quas (University of Victoria) who heard it, and the solution below, from Isaac Kornfeld, a professor at Northwestern University. Kornfeld had heard the puzzle many years ago in Moscow. It appeared later in the International Mathematics Tournament of the Towns, as Problem 5 in the Fall 1998 Senior A-Level Paper (but with a different solution).

The two-dimensional version of the puzzle is easily solved using the triangle inequality, but the third dimension appears to require some new idea. Let B_ε be the expansion, by an amount $\varepsilon > 0$, of a box B ; that is, the set of all points in space within distance ε of some point of B . If B is $a \times b \times c$ then B_ε will be approximately $(a+2\varepsilon) \times (b+2\varepsilon) \times (c+2\varepsilon)$, but with rounded edges and corners. The precise volume of B_ε will be abc (the volume of B) plus $2ab\varepsilon + 2ac\varepsilon + 2bc\varepsilon$ (the volume of the slabs added to the six sides) plus $4a\pi\varepsilon^2/4 + 4b\pi\varepsilon^2/4 + 4c\pi\varepsilon^2/4$ (the volume of the 12 "molding strips" added to the edges—each having a quarter-circle cross section—plus $4\pi\varepsilon^3/3$, since the 8 knobs added to the corners add up to a sphere. Altogether,

$$\text{Vol}(B_\varepsilon) = \frac{4}{3}\pi\varepsilon^3 + (a+b+c)\pi\varepsilon^2 + 2(ab+ac+bc)\varepsilon + abc .$$

²The cell probe complexity of succinct data structures, ICALP 2003.

Now, if box A (with dimensions, say, a' , b' and c') lies inside box B , then A_ε lies inside B_ε , for any $\varepsilon > 0$. Hence $\text{Vol}(A_\varepsilon) < \text{Vol}(B_\varepsilon)$. But, if we take ε to be *huge*, the dominant term in the difference of their volumes is

$$(a+b+c)\pi\varepsilon^2 - (a'+b'+c')\pi\varepsilon^2.$$

Since this term must be non-negative, B is the costlier box.

10 solution to The Dot-town Suicides

This puzzle came to me from Nick Reingold, of AT&T Labs. Various more specific versions (often in even poorer taste than this one) have existed for many decades.

The proof (my own) of this very general version uses backwards induction on the number of disallowed numbers of blue dots. In the base case there are $n-1$ such numbers, where n is the population of Dot-town. In that case everyone can immediately deduce his own dot-color, so the town is wiped out immediately.

Say there are n residents and the stranger announces that the number of blue dots does not belong to the set K , where K is some non-empty subset of the numbers from 0 to n .

Suppose the actual number of blue dots is j , so that the red-dotted residents see j blue dots and the others see only $j-1$. If $j-1$ is in K then the blue-dotted residents will all kill themselves the first night. The remaining residents (if any) will deduce that their spots are red, and dispense with themselves the following night.

If $j+1$ is in K then the red-dotted residents will all kill themselves the first night, and the remaining residents will follow suit the next night after deducing that their spots are blue.

If no one kills himself the first night, everyone can deduce that neither $j-1$ nor $j+1$ is in K , i.e. the actual number of blue dots is not within one of any number in K . This increases the number of forbidden numbers of blue dots, and the induction hypothesis can now be applied.

Note that the proof shows that n nights suffice; further, that the full n nights are needed only when $K = \{0\}$ or $\{n\}$ and either all dots are the same color or $n \leq 2$.

11 solution to Unwanted Expansion

This curious puzzle was communicated to me by Dick Lipton, of the College of Computing at Georgia Tech.

One can analyze the expression in terms of depth of trees, but there's an easier way: set all the variables equal to 2! The point of the distributive law is that its application doesn't change the value of the expression. The value of the initial expression limits the size of anything you can get from it by expansion.

12 solution to Love in Kleptopia

This puzzle came from Caroline Calderbank, young daughter of mathematicians Ingrid Daubechies and Rob Calderbank. In the solution she had in mind, Jan sends Maria a box with the ring in it and one of his padlocks on it. Upon receipt Maria affixes her own padlock to box and mails it back with both padlocks on it. When Jan gets it he removes his padlock and sends the box back to Maria; voila! This solution is not just play; the idea is fundamental in Diffie-Hellman key exchange, an historic breakthrough in cryptography.

Depending on one's assumptions, other solutions are possible as well. My favorite was suggested by several persons at the Gathering for Gardner VII: it requires that Jan find a padlock whose key has a large hole, or at least a hole which can be sufficiently enlarged by drilling, so that the key can be hooked onto a second padlock's hasp.

Jan uses this second padlock, with the aforementioned key hooked on its hasp, to lock a small empty box which he then sends to Maria. When enough time has passed for it to get there (perhaps he awaits an email acknowledgment from Maria) he sends the ring in another box, locked by the

first padlock. When Maria gets the ring box, she picks up the whole first box and uses the key affixed to it to access her ring.

13 solution to How to Win at Wimbledon

It sounds obvious that you should ask to be ahead two sets to love (it takes 3 out of 5 sets to win the men's), and in the third set, ahead 5-0 in games and 40-love in the sixth game. (Probably you want to be serving, but if your serve is like mine, you might prefer Roger to be serving the sixth game down 0-40 so that you can pray for a double fault.)

Not so fast! These solutions give you essentially 3 chances to get lucky and win, but you can get six chances—with three services by you and three by Roger. You still want to be up two sets to none, but let the game score be 6-6 in the third set and 6-0—in your favor, of course—in the tiebreaker.

Amit Chakrabarti of Dartmouth suggested the following improvement, based on the idea that traditionally the complete score of a tennis match includes the game scores of all sets and, if the game score was 6-6, the tiebreaker score as well. Then you could ask for example that the score be 6-0, 6-6 (9999-9997), 6-6 (6-0). The theory here (dubious, granted) is that while you were under your magic spell, Roger was wearing himself out in the second-set tiebreaker and is now more likely to blow one of the six upcoming match points.

14 solution to The Random Native

This version of “logician at the crossroads” came to me from two mathematical physicists, Vldas Sidoravicius and Senya Shlosman. It seems impossible to deal with the random native, but you can.

The idea is, you need to be sure the *second* native you query is not the random answerer. This is necessary because you're not going to know the road after the first question, and if the second responder is random you won't learn any more.

On the other hand, this objective is sufficient, because you can then use the traditional one-native query as your second question: namely, something like “If I were to ask you whether this road goes to the village, would you say yes?”.

To attain the objective, you'll need to ask Native *A* something about Native *B* or Native *C*, then use the answer to choose between *B* and *C*. Here's one that works: “Is *B* more likely than *C* to tell the truth?”

Curiously, if *A* says “yes” you pick *C*, and if he says “no” you pick *B*! If *A* is the truth-teller you want next to query the companion who is *less* likely to tell the truth, namely, the liar. If *A* is the liar you want the *more* truthful of his companions, namely the truth-teller.

Of course, if *A* is the random answerer it doesn't matter which of *B* and *C* you turn to next.

In Martin Gardner's columns it was pointed out that in the original one-native problem, the logician can get to the village even if he has forgotten which of the native words (“pish” and “tush”) means “yes” and which means “no.” Readers seeking further challenge can attempt to modify the above protocol similarly.

Acknowledgment

Inspiration for collecting these puzzles and much more comes from Martin Gardner's famous *Mathematical Games* column in *Scientific American*. Thanks also to my (former) friends who served as test victims for countless puzzles.