Brill–Noether special cubic fourfolds of discriminant 26

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AMS Spring Central Sectional April 16th, 2023 Let $\mathcal C$ be the 20-dimensional moduli space of smooth cubic fourfolds $X\subset \mathbb P^5.$

Let X be a cubic fourfold, we denote by

$$A(X) := H^{2,2}(X) \cap H^4(X, \mathbb{Z})$$

the lattice of algebraic cycles.

Definition

X is special if $\operatorname{rk} A(X) \ge 2$.

That is, X contains an algebraic surface T not homologous to a complete intersection.

The special cubic fourfolds are a union of divisors Hassett C_d of cubic fourfolds such that A(X) admits a primitive embedding preserving h^2 of a lattice of discriminant d.

 C_d is nonempty if d > 6 and $d \equiv 0$ or 2 (mod 6). (e.g., 8, 12, 14, 18, 20, 24, 26, ...)

More generally, one can consider *lattice polarized* cubic fourfolds.

Definition

Let Λ be a positive definite lattice with a distinguished element $h^2 \in \Lambda$ of norm 3.

A Λ -polarized cubic fourfold, (X, Λ) , is a cubic fourfold X together with a fixed primitive isometric embedding $\Lambda \hookrightarrow A(X)$ preserving h^2 .

Let \mathcal{C}_Λ denote the moduli space of $\Lambda\text{-polarized}$ cubic fourfolds.

By work of Laza and Looijenga, C_{Λ} is non-empty if and only if Λ admits a primitive embedding into $H^4(X, \mathbb{Z})$ and Λ has no *short roots* or *long roots*.

Let Λ be a positive definite lattice with distinguished element h^2 of norm 3.

Definition

We say $v\in \langle h^2\rangle^{\perp}\subseteq \Lambda$ is a

• short root if $v^2 = 2$

• long root if
$$v^2 = 6$$
 and $v \pm h^2 \in 3\Lambda$.

A positive definite lattice Λ such that \mathcal{C}_Λ is non-empty is called a *cubic fourfold lattice*.

Following Hassett's argument for C_d shows that C_Λ is a $(21 - \mathrm{rk} \Lambda)$ -dimensional quasiprojective variety.

Let
$$d>6$$
 divide $2(n^2+n+1)$, (e.g. 14, 26, 38, 42, ...)

For a cubic fourfold X with a marking of discriminant d, there is an associated polarized K3 surface (S, H) of genus $g = \frac{d}{2} + 1$ such that there is a Hodge isometry

$$H^4(S,\mathbb{Z}) \supset K_d^{\perp} \cong H^{\perp}(-1) \subset \operatorname{Pic}(S)(-1).$$

This gives rise to an open immersion $\mathcal{C}_{\Lambda} \hookrightarrow \mathcal{K}_g$ of moduli spaces.

We extend this open immersion to Λ -polarized cubic fourfolds.

Lattice theory

Let

$$K_{d} = \begin{array}{cccc} h^{2} & T \\ T & 3 & x \\ T & x & y \end{array} \subset \Lambda = \begin{array}{cccc} h^{2} & T & J \\ 3 & x & a \\ T & x & y & b \\ J & a & b & c \end{array}$$

with K_d be a positive definite lattice of discriminant d and Λ a cubic fourfold lattice.

Lemma

Suppose $gcd(3, h^2.T)|h^2.J$ and $-3 \in (\mathbb{Z}/d\mathbb{Z})^{\times 2}$. Then, up to isometry, there is a unique rank 2 even indefinite lattice

$$\sigma(\Lambda) = \begin{array}{cc} H & M \\ \hline d & \alpha \\ M & \alpha & \beta \end{array}$$

with discriminant $-\operatorname{disc}(\Lambda)$ such that $K_d^{\perp} \cong H^{\perp}(-1)$.

Corollary

Since $gcd(3, h^2.T)|h^2.J$, we can choose a = 0, i.e., there is a basis of Λ with respect to which Λ has Gram matrix

for some integers $0 \le b \le d/2$ and $c > \max(2, 3b^2/d)$ even.

$\operatorname{Pic}(S)$ of associated K3

The standard form of Λ gives us an algorithm for computing $\sigma(\Lambda)$.

For fixed K_d ,

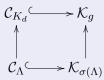
$$\Lambda = \begin{array}{cccc} h^2 & T & J \\ 3 & x & 0 \\ T & x & y & b \\ J & 0 & b & c \end{array}$$
 is determined by (b, c) .

$$\sigma(\Lambda) = \begin{array}{ccc} H & M \\ H & d & \alpha \\ M & \alpha & \beta \end{array}$$
 is determined by (α, β) .

We can recover Λ with the fixed embedding of K_{26} from $\sigma(\Lambda)$.

Theorem (Auel–H.)

Let d = 2g - 2 have a unique odd prime divisor. Let $\Lambda = \langle h^2, T, J \rangle$ be a rank 3 cubic fourfold lattice with a fixed primitive embedding of K_d preserving h^2 such that $gcd(3, h^2.T)|h^2.J$. Then there exists an open immersion $C_{\Lambda} \hookrightarrow \mathcal{K}_{\sigma(\Lambda)}$ of moduli spaces and a commutative diagram



where the vertical arrows are the forgetful maps and the top arrow is the open immersion constructed by Hassett.

Cubic fourfolds of discriminant 26

$$K_{26} = \begin{array}{c} h^2 & T \\ \hline 3 & 7 \\ T & 7 & 25 \end{array}$$

A cubic fourfold containing a 3-nodal septic scroll is K_{26} -polarized, as does the general $X \in C_{26}$.

We focus on cubic fourfolds that are Λ -polarized, where Λ has a fixed primitive embedding $K_{26} \hookrightarrow \Lambda$.

{What does Λ look like?} \iff {What does $\operatorname{Pic}(S)$ look like?}

Definition (Mukai)

A polarized K3 surface (S, H) of genus g, $(H^2 = 2g - 2)$, is Brill–Noether special if there is a nontrivial $J \neq H \in Pic(S)$ such that

$$g - h^0(S, J)h^0(S, H - J) < 0.$$

Else (S, H) is called *Brill–Noether general*.

Definition (Auel)

A special cubic fourfold X is *Brill–Noether special* if it has an associated polarized K3 surface (S, H) that is Brill–Noether special.

Definition

A curve C of genus g is ${\it Brill-Noether\ special}$ if it has a line bundle A such that

$$p(g,r,d) = \underbrace{g}_{\mathsf{genus}(C)} - \underbrace{(r+1)}_{h^0(C,A)} \underbrace{(g-d+r)}_{h^0(C,\omega_C-A)} < 0.$$

In genus 8, a curve is BN special if and only if it has a g_7^2 . In higher genus, this becomes more complicated.

Theorem (Auel–H.)

Let (S, H) be a polarized K3 surface of genus 14. Then (S, H) is Brill–Noether special if and only if a smooth irreducible curve $C \in |H|$ is Brill–Noether special.

Theorem (Mukai, Lelli-Chiesa, Auel)

A cubic fourfold of discriminant 14 is Brill–Noether special if and only if the associated K3 satisfies that $C \in |H|$ has a g_7^2 . Moreover, such cubic fourfolds contain 2 disjoint planes.

Proposition (Auel)

A cubic fourfold is pfaffian if and only if it has a discriminant 14 marking whose associated K3 is Brill–Noether general.

Theorem (Auel)

The complement of the pfaffian locus is contained in the locus of cubic fourfolds containing 2 disjoint planes.

Lattices of Brill–Noether special K3s of genus 14

Let \mathcal{K}_g be the moduli space of primitively quasi-polarized K3 surfaces of genus g.

The Noether–Lefschetz divisor $\mathcal{K}_{g,d}^r \subset \mathcal{K}_g$ parameterizes K3 surfaces with a specific lattice polarization

$$\Lambda_{g,d}^r := \begin{array}{cc} H & L \\ \hline 2g-2 & d \\ L & d & 2r-2 \end{array} \subseteq \operatorname{Pic}(S).$$

Theorem (Greer-Li-Tian)

(S,H) is Brill–Noether special if and only if $(S,H) \in \mathcal{K}^r_{q,d}$ where

• $2 \le d \le g - 1$,

•
$$\Delta(g, r, d) := \operatorname{disc}(\Lambda_{g, d}^r) = 4(r - 1)(g - 1) - d^2 < 0$$
,

• and $\rho(g, r, d) < 0$.

For each of these lattices, we recover a lattice Λ such that $\sigma(\Lambda) = \Lambda^r_{g,d}$, excluding those with roots.

Theorem (Auel–H.)

Let X be a Brill–Noether special cubic fourfold of discriminant 26, and (S, H) the associated polarized K3 of genus 14. Then $\operatorname{Pic}(S)$ has a primitive embedding of one of the following lattices:

•
$$\Lambda^1_{14,6}$$
 $(\gamma(C) = 4)$

•
$$\Lambda^2_{14,9}$$
 $(\gamma(C) = 5)$

•
$$\Lambda^2_{14,10}$$
 ($\gamma(C) = 6$)

•
$$\Lambda^2_{14,11}$$
, $\Lambda^3_{14,13}$ $(\gamma(C) = 6)$

In particular, for the general such K3 surface, $\gamma(C) = 4, 5$, or 6, and C has no other Brill–Noether special line bundles of Clifford index $\leq \gamma(C)$.

Interesting Overlaps

Given $\Lambda^r_{g,d}=\sigma(\Lambda)$ on the list above, we look for which other markings Λ admits.

 $\Lambda^1_{14,6}$

	Η	L				T	~
	11	<u></u>	gives the lattice	h^2	3	7	0
Н	26	6		T	7	25	12
L	6	0		J	0	12^{-3}	18

•
$$\langle h^2, -2h^2 + T - J \rangle$$
 gives a discriminant 8 marking.

• $\langle h^2, 6h^2 - 2T + J \rangle$ gives a discriminant 14 marking.

We note that such cubic fourfolds form a component of $C_8 \cap C_{14}$, due to Auel–Bernardara–Bolognesi–Várilly-Alvarado, and the general one has non-trivial Clifford invariant $\beta \in Br(S)$.

 $\Lambda^2_{14,9}$

$$\begin{array}{c|cccccc} H & L \\ \hline H & 26 & 9 \\ L & 9 & 2 \end{array} \text{ gives the lattice } \begin{array}{c|cccccccccc} h^2 & T & J \\ h^2 & 3 & 7 & 0 \\ T & 7 & 25 & 5 \\ J & 0 & 5 & 4 \end{array}$$

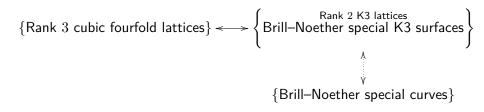
•
$$\langle h^2, -2h^2 + T - J \rangle$$
 gives a discriminant 8 marking.
• $\langle h^2, -h^2 + T - 2J \rangle$ gives a discriminant 14 marking.

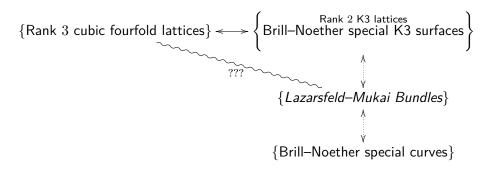
We note that such cubic fourfolds form a component of $C_8 \cap C_{14}$, due to Auel–Bernardara–Bolognesi–Várilly-Alvarado, and have trivial Clifford invariant, hence are rational.

Theorem (Auel–H.)

Let X be a Brill–Noether special cubic fourfold of discriminant 26, then $X \in C_8$.

We note that general Brill–Noether special cubic fourfolds of discriminant 26 admit no C_{42} marking.





Thank You!

Questions?