Computations with Affine Connections

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May 6, 2022

Outline

An Affine connection on a manifold is used to describe the notion of *parallel translation*. The definition of such an object is motivated by the *covariant derivative* from the calculus of $\mathbb{R}^n$. We'll use this to compute the angle of deviation of a pendulum in Paris swinging north-to-south over a 24 hour period.

Basic Definitions

A *section* of a manifold to it's tangent bundle is a function $f : M \to TM$ such that for all $p \in M$ we have $\mathrm{proj}\big(f(p)\big) = p$, where $\mathrm{proj}\big((x, v)\big) = x$, $(x, v) \in TM$. A smooth vector field on a smooth manifold $(M, \mathcal{A})$ is a smooth section $X : M \to TM$.

An equivalent formulation is an assignment at each $p \in M$ a tangent vector $X_p \in T_pM$ and this assignment is done *smoothly*. That is, for any given function $f \in C^\infty(M, \mathbb{R})$ the function Xf, defined in a local chart $(\mathcal{U}, \varphi)$ as:

$$Xf(p) = X_pf = \sum_{n=0}^{N-1} a_n \frac{\partial f}{\partial \varphi_n}(p) \tag{1}$$

is smooth.

Basic Definitions

Since a vector field $X : M \to TM$ can be applied to smooth functions $f \in C^\infty(M, \mathbb{R})$, the result being a smooth, it is possible to compose vector fields $X, Y : M \to TM$. That is, given a function $f \in C^\infty(M, \mathbb{R})$ and two smooth vector fields X and Y, we obtain a new smooth function $X(Yf)$. The composition of two vector fields need not be a vector field. Using local coordinates $(\mathcal{U}, \varphi)$, we may represent the tangent vectors X_p and Y_p as follows:

$$X_p = \sum_{n=0}^{N-1} a_n(p) \frac{\partial}{\partial \varphi_n} \tag{2}$$

$$Y_p = \sum_{n=0}^{N-1} b_n(p) \frac{\partial}{\partial \varphi_n} \tag{3}$$

Basic Definitions

The composition $X(Yf)$ is then:

$$X\Big(\sum_{m=0}^{N-1} b_n \frac{\partial f}{\partial \varphi_m}\Big) = \sum_{n=0}^{N-1} a_n \frac{\partial}{\partial \varphi_n}\Big(\sum_{m=0}^{N-1} b_n \frac{\partial f}{\partial \varphi_m}\Big) \tag{4}$$

$$= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\Big(a_n \frac{\partial b_m}{\partial \varphi_n}\frac{\partial f}{\partial \varphi_m} + a_n b_m \frac{\partial^2 f}{\partial \varphi_n \partial \varphi_m}\Big) \tag{5}$$

So XYf involves second order derivatives, which is not Liebnizian. If we subtract YXf and invoke the Clairaut formula, we see that $XY - YX$ involves only first order derivatives, and hence is a vector field. This is called the *Lie Bracket* of X with respect to Y, $[X, Y] = XY - YX$.

Basic Definitions

A Riemannian metric on a smooth manifold $(M, \mathcal{A})$ is a function g on M such that for all $p \in M$, g_p is a symmetric bilinear form that is positive-definite. That is, for all $u_0, u_1, v_0, v_1 \in T_pM$ and $a, b, c, d \in \mathbb{R}$ we have:

$$g_p(u_0,\, u_1) = g_p(u_1,\, u_0) \tag{6}$$

$$\begin{aligned} g_p(au_0 + bu_1,\, cv_0 + dv_1) = {} & acg_p(u_0,\, v_0) + adg_p(u_0,\, v_1) + \\ & bcg_p(u_1,\, v_0) + bdg_p(u_1,\, v_1) \end{aligned} \tag{7}$$

$$g_p(u_0,\, u_0) \geq 0 \tag{8}$$

$$g_p(u_0,\, u_0) = 0 \quad \text{iff} \quad u_0 = 0 \tag{9}$$

Basic Definitions

Moreover, the function g should vary *smoothly* with p. That is, for any two smooth vector fields X and Y, the function $f : M \to \mathbb{R}$ defined by:

$$f(p) = g_p(X_p, Y_p) \tag{10}$$

should be smooth. Such a function g is called a Riemannian metric, and a Riemannian manifold is an ordered triple $(M, \mathcal{A}, g)$ where $(M, \mathcal{A})$ is a smooth manifold and g is a Riemannian metric on $(M, \mathcal{A})$.

Basic Definitions

The *covariant* derivative in $\mathbb{R}^n$ gives us a way of specifying the *derivative* of one vector field with respect to another. Given $X = \sum a_n(\mathbf{x})\partial x_n$ and $Y = \sum b_m(\mathbf{x})\partial x_m$, the covariant derivative of Y with respect to X is:

$$\sum_{n=0}^{N-1} \sum_{m=0}^{N-1} a_n \frac{\partial b_m}{\partial x_n} \frac{\partial}{\partial x_m} \tag{11}$$

This is the part of XY that does not involve second order terms.

Affine Connections

An affine connection of a smooth manifold $(M, \mathcal{A})$ axiomatizes the properties of the covariant derivative. This is a function $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the set of all smooth vector fields on $(M, \mathcal{A})$, with the following properties:

- ∇ is bilinear.
- ∇ is $C^\infty(M, \mathbb{R})$ linear in the first coordinate:

$$\nabla_{fX} Y = f \nabla_X Y \tag{12}$$

- ∇ is Liebnizean in the second coordinate:

$$\nabla_X fY = (Xf)Y + f \nabla_X Y \tag{13}$$

Recall that since X is a smooth vector field, Xf is a smooth function, so $(Xf)Y$ is the product of a smooth function with a smooth vector field, which is again a smooth vector field. The third condition is thus the sum of two vector fields, which is a vector field, meaning all of this is well defined.

Affine Connections

All of this is defined for smooth manifolds and no Riemannian metric is yet needed. Affine connections are called *torsion free* if they are related to the Lie bracket:

$$\nabla_X Y - \nabla_Y X = [X, Y] \tag{14}$$

Given a Riemannian manifold $(M, \mathcal{A}, g)$, the connection is said to be *compatible* with g if for all smooth vector fields X, Y, Z we have

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \tag{15}$$

A *Levi-Civita* connection is an affine connection that is torsion free and is compatible with the metric g.

Affine Connections

Theorem (Fundamental Theorem of Riemannian Geometry)

Every Riemannian manifold $(M, \mathcal{A}, g)$ *has a unique Levi-Civita connection* ∇.

Parallel Transport

Given a smooth curve $\gamma : [0,\ 1] \to M$, if γ is injective, then the image $\gamma\big[[0,1]\big]$ defines a submanifold with boundary in M. The velocity vector $\dot{\gamma}(t_0)$ is defined as the derivation $\dot{\gamma}(t_0) : C^{\infty}(M, \mathbb{R}) \to \mathbb{R}$ given by:

$$\dot{\gamma}(t_0)(f) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(f \circ \gamma\Big) \tag{16}$$

Note that $f \circ \gamma$ is a function from $[0, 1]$ to $\mathbb{R}$ so we may differentiate in the usual sense. $\dot{\gamma}$ defines a vector field on a closed submanifold with boundary of M. It is always possible to extend a smooth vector field on a closed submanifold to a smooth vector field on all of M. This follows from a partition of unity argument.

Parallel Transport

Theorem

If $(M, \mathcal{A})$ is a smooth manifold, if ∇ is an affine connection on M, if γ is a smooth injective curve in M, if X and Y are smooth extensions of $\dot{\gamma}$, and if Z is a smooth vector field, then for all $p \in M$ such that $p = \gamma(t)$ for some $t \in [0, 1]$, we have:

$$\nabla_{X_p} Z_p = \nabla_{Y_p} Z_p \tag{17}$$

Because of this we may abuse notation and write $\nabla_{\dot{\gamma}} Z$ to mean the derivative of the vector field Z along the curve γ.

Parallel Transport

A vector field that is *parallel* along a curve γ is a smooth vector field X such that $\nabla_{\dot{\gamma}} X = 0$. Given a curve γ and a tangent vector $v \in T_{\gamma(0)}M$ we can *solve* for a vector field that is parallel along γ in local coordinates by solving a system of differential equations. This allows one numerically solve for how a given tangent vector will be transported along a curve.

The Foucault Pendulum

Let's use these ideas on $\mathbb{S}^2$.

Given a Riemannian manifold $(M, \mathcal{A}, g)$ and a smooth embedding $f : X \to M$ of a smooth manifold $(X, \mathcal{A}_X)$ it is possible to endow $(X, \mathcal{A}_X)$ with a metric via *pull-back*. Give $x \in X$, $u_0, u_1 \in T_x X$, let $p = f(x)$ and $v_0 = \mathrm{d}f_x(u_0)$, $v_1 = \mathrm{d}f_x(u_1)$, where $\mathrm{d}f_x$ is the differential push-forward of f at the point x. We may define $\tilde{g}_x$ via:

$$\tilde{g}_x(u_0, u_1) = g_p(v_0, v_1) \tag{18}$$

Since the differential push-forward of a smooth vector field is a smooth vector field, if g is smooth, then so is $\tilde{g}$. This gives $(X, \mathcal{A}_X, \tilde{g})$ the structure of a Riemannian manifold.

The Foucault Pendulum

The round metric on $\mathbb{S}^2$ is obtained via pull-back. $\mathbb{R}^3$ has the standard dot product:

$$\langle \mathbf{x} | \mathbf{y} \rangle = \sum_{n=0}^{N-1} x_n y_n \tag{19}$$

The inclusion mapping $\iota : \mathbb{S}^2 \to \mathbb{R}^3$ is a smooth embedding, and hence induces a metric on $\mathbb{S}^2$ via pull-back of the dot product. This Riemannian metric on $\mathbb{S}^2$ is called the *round metric*.

The Foucault Pendulum

There is a correspondence between vector fields on $\mathbb{S}^2$ and functions $X : \mathbb{S}^2 \to \mathbb{R}^3$ such that for all $p \in \mathbb{S}^2$ we have $\langle p | X_p \rangle = 0$. A smooth vector field is in particular a smooth function $X : \mathbb{S}^2 \to \mathbb{R}^3$ with this property. The function $\nabla : \mathfrak{X}(\mathbb{S}^2) \times \mathfrak{X}(\mathbb{S}^2) \to \mathfrak{X}(\mathbb{S}^2)$ defined by:

$$\nabla_{X_p} Y_p(f) = \mathrm{d}Y_p(X_p)(f) + \langle X_p | Y_p \rangle f(p) \tag{20}$$

determines a Levi-Civita connection on $\mathbb{S}^2$.

The Foucault Pendulum

Now imagine it is 1851, you are in Paris, and have a 67 meter long pendulum attached to the dome of the French Panthéon. You have it swinging north-to-south. You then ask what will the angle be after one full rotation of the Earth.

It is easy to convince your self that if you were in Brazil on the equator and performed this experiment, the angle would not deviate at all. It is also easy to believe that if you performed this experiment next to polar bears at the north pole the pendulum would simply rotate with the Earth a full 2π radians.

The Foucault Pendulum

The path the pendulum traverses is a circle of constant latitude. We can cover this in a single coordinate chart via stereographic projection about either the north or south pole. In these coordinates x and y correspond to east-west and south-north directions, respectively. The system of differential equations we need to solve is:

$$\ddot{x}(t) = -x + 4\pi\dot{y}(t)\sin(\phi) \tag{21}$$

$$\ddot{y}(t) = -y - 4\pi\dot{x}(t)\sin(\phi) \tag{22}$$

Here ϕ is the angle of latitude of Paris. This converts into a single complex differential equation:

$$\ddot{z}(t) + 4\pi i\sin(\phi)\dot{z}(t) + z(t) = 0 \tag{23}$$

The Foucault Pendulum

We can solve directly:

$$z(t) = \exp\Big(-2\pi i \sin(\phi) t\Big)\Big(A\exp(it) + B\exp(-it)\Big) \qquad (24)$$

where A and B are constants corresponding to the initial tangent vector v (which is north-south). Here, t is measured in days. The angle of deviation after one rotation of the Earth is then $-2\pi \sin(\phi)$. The latitude of Paris is 48 degrees and 51 minutes, or about 0.85 radians. The resulting deviation is -4.73, or about -271.1 degrees. The pendulum will be oscillating in the east-west direction.