# Riemannian and Semi-Riemannian Manfifolds

## Similarities and Differences

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A smooth manifold is a locally Euclidean Hausdorff topological space that is second countable such that there is an atlas $\mathcal{A}$ consisting of smoothly compatible charts.

Diffeomorphisms are smooth bijective functions with smooth inverses.

Things like *distance* and *volume* are not preserved by diffeomorphisms, indeed these concepts aren't even well-defined for the general smooth manifold.

A smooth vector field on a manifold $M$ is a smooth section of the tangent bundle $X : M \to TM$.

Let's define a *generalized metric* to be a function $g$ on $M$ such that for all $p \in M$, $g_p : T_pM \to \mathbb{R}$ is a smooth symmetric bilinear form. By smooth it is meant that for all smooth vector fields $X, Y : M \to TM$ the function $g_p(X_p, Y_p)$, which is a function from $M$ to $\mathbb{R}$, is smooth.

At this point there is no requirement of positive definiteness or non-degeneracy.

A *Riemannian* metric on a manifold $M$ is a generalized metric $g$ that is positive-definite. That is, for all $p \in M$ and for all $v \in T_pM$, $g_p(v, v) \geq 0$ and $g_p(v, v) = 0$ if and only if $v$ is the zero tangent vector.

A *semi-Riemannian* metric is a generalized metric that is non-degenerate. That is, for all $p \in M$ and all non-zero $v \in T_pM$, there is a $w \in T_pM$ such that $g_p(v, w) \neq 0$.

By Sylvester's Law of Inertia (I have absolutely no idea why it is called that), for each $p \in M$ there is a basis of $T_pM$ such that the matrix representation of the symmetric bilinear form $g_p$ is diagonal and consists entirely of 1, 0, and $-1$ on the diagonal.

By Sylvester's Conservation of Inertia (again, no idea why the name), the number of 1's, 0's, and $-1$'s is a constant. This allows one to define the *signature* of a generalized metric. This is the ordered triple $(a, b, c)$ where $a \in \mathbb{N}$ is the number of 1's in such a representation, $b \in \mathbb{N}$ is the number of 0's, and $c \in \mathbb{N}$ is the number of $-1$'s.

An $N \in \mathbb{N}$ dimensional Riemannian manifold is just a smooth manifold with a generalized metric of signature $(N, 0, 0)$. A semi-Riemannian manifold is a smooth manifold with a generalized metric of signature $(n, 0, m)$ with $n + m = N$, $n, m \geq 0$.

Many of the results about Riemannian manifolds hold for semi-Riemannian, and many do not not. In this talk we'll discuss some of these central ideas.

Note, since a semi-Riemannian metric $g$ is non-degenerate by definition, the number $b$ in the signature $(a, b, c)$ of $g$ is always zero. Because of this many authors write the signature of $g$ to be $(a, c)$.

An affine connection on a smooth manifold is a function $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that:

- $\nabla$ is bilinear.
- $\nabla$ is $C^\infty(M, \mathbb{R})$ linear in the first component. That is, $\nabla_{fX} Y = f \nabla_X Y$.
- $\nabla$ is Liebnizean in the second component. That is, $\nabla_X fY = (Xf)Y + f\nabla_X Y$.

A torsion-free affine connection is one compatible with the *Lie Bracket*. That is:

$$\nabla_X Y - \nabla_Y X = [X, Y] \tag{1}$$

Given a generalized metric $g$ on a smooth manifold $M$, a *compatible* affine connection is an affine connection $\nabla$ such that for all smooth vector fields $X, Y, Z$ we have:

$$Xg_p(Y_p, Z_p) = g_p(\nabla_X Y, Z) + g_p(Y, \nabla_X Z) \tag{2}$$

A Levi-Civita connection on a smooth manifold with a (generalized) metric $g$ is an affine connection that is torsion free and compatible with $g$.

A theorem dating back to the early 1900's states if $M$ is a smooth manifold with a *Riemannian* metric, then there is a unique Levi-Civita connection on $M$. This generalizes to the semi-Riemannian case.

**Theorem (Fundamental Theorem of Semi-Riemannian Geometry)**

*If $M$ is a smooth manifold, and if $g$ is a semi-Riemannian metric, then there is a unique Levi-Civita connection on $M$.*

This does not generalize to generalized metrics, the non-degenaracy is needed. Indeed, take $g$ to be the *zero* metric with signature $(0, N, 0)$. Then *any* affine connection is compatible with $g$ since the compatibility equation reduces to $0 = 0 + 0$, which is true.

For both Riemannian and semi-Riemannian the fundamental theorem generalizes as follows. Define any function $F : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$. We have the following.

### Theorem (Generalized Fundamental Theorem of Semi-Riemannian Geometry)

*If $M$ is a smooth manifold with a semi-Riemannian metric $g$, then there is a unique affine connection that is compatible with $g$ such that for all $X, Y \in \mathfrak{X}(M)$ we have:*

$$\nabla_X Y - \nabla_Y X - [X, Y] = F(X, Y) \tag{3}$$

*That is, there is a unique affine connection compatible with $g$ with the prescribed torsion.*

The proof is the same as the proof of the previous theorem. One takes the *Koszul formula*, shows that any such metric mush satisfy it, and that the equation does indeed define an affine connection compatible with $g$.

Affine connections can be defined locally on curves, and need not be defined on the entirety of $M$. A smooth curve $\gamma$ in $M$ is called a *geodesic* with respect to $\nabla$ if:

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0 \tag{4}$$

This makes sense for Riemannian, semi-Riemannian, or generalized metrics, meaning there is always a way to define geodesics.

What differs is the use of geodesics to define a *distance function* (a regular metric-space metric) on $M$ induced by $g$. For Riemannian metrics $g$ one may define:

$$d(p,q) = \begin{cases} \inf_{\gamma : p \to q} \int_\gamma \sqrt{g_{\gamma(t)}(\dot{\gamma(t)}, \dot{\gamma(t)})} \mathrm{d}t & p \text{ connected to } q \\ 1 & \text{else} \end{cases} \tag{5}$$

Positive-definiteness shows that this is well-defined, and with a bit of work one can show this transforms $(M, d)$ into a metric space.

It is not so easy to extend this idea to semi-Riemannian manifolds, at least without modification. Take a *Lorentz manifold*, which is a semi-Riemannian manifold with metric $(N-1,0,1)$. Given $p \in M$ there are points $v \in T_pM$ such that $g_p(v,v)=0$ but $v \neq 0$. The set of all such points is called the *light-cone* of $p$ since the set of points satisfies the equation:

$$\Big(\sum_{n=0}^{N-2} \mathrm{d}x_n^2\Big) - \mathrm{d}t^2 = 0 \tag{6}$$

Solving for $\mathrm{d}t$ as a function of the other one-forms $\mathrm{d}x_n$ gives the equation of a cone. Abusing notation, it is the *light-cone* since we have:

$$\sum_{n=0}^{N-2}\Big(\frac{\mathrm{d}x_n}{\mathrm{d}t}\Big)^2 = 1 \tag{7}$$

That is, the velocity vector has norm 1 which in *natural units* corresponds to the speed of light.

The formula for distance between points does not work with a Lorentzian metric. There are vectors $v \in T_pM$ with $g_p(v, v) < 0$, the so-called *time-like* vectors which represent movement at less than the speed of light. The square root of $g_{\gamma(t)}(\dot{v}(t), \dot{v}(t))$ is not a real number so this formula does not make sense. Moreover, when it does make sense it may not define a metric. Points that differ by a light-like curve (a curve with $g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)) = 0$) will have a *distance* between them of zero.

Another celebrated theorem of Riemannian geometry does not hold in semi-Riemannian geometry.

**Theorem (Hopf-Rinow Theorem)**

*If $(M, g)$ is a connected Riemannian manifold, then the following are equivalent ($d$ being the metric induced by $g$):*

1. *Closed bounded subsets of $M$ are compact.*
2. *$(M, d)$ is a complete metric space.*
3. *$M$ is geodesically complete.*

Geodesically complete means geodesics may flow for all time. This theorem is *false* for (and not well-posed) for semi-Riemannian manifolds. However, even if we omit the second statement, conditions 1 and 3 are not equivalent in a semi-Riemannian manifold.

Let $M = \mathbb{R}^2 \setminus \{(0,0)\}$. Define $g$ on $M$ as follows:

$$g = 2\frac{\mathrm{d}x\,\mathrm{d}y}{x^2 + y^2} \tag{8}$$

The function $\lambda : M \to M$ defined by $\lambda((x,y)) = (2x, 2y)$ is an isometry. Let $\Gamma$ be the subgroup of the isometry group of $(M, g)$ generated by $\lambda$. There is a properly discontinuous group action of $\Gamma$ on $M$ and the space $M/\Gamma$ is, topologically, the torus $\mathbb{T}^2$. In particular, $M/\Gamma$ is compact and hence *any* metric on $M/\Gamma$ that induces it's topology must be bounded. The induced metric gives a *Lorentz surface*, and this ordered pair $(M/\Gamma, \tilde{g})$ is called the *Clifton-Pohl* torus.

The Clifton-Pohl torus is *not* geodesically complete, even though it is compact. The geodesic:

$$\gamma(t) = \left(\frac{1}{1-t}, 0\right) \tag{9}$$

in $M$ induces a geodesic $\tilde{\gamma}$ in $M/\Gamma$, but this induced curve cannot flow for time $t \geq 1$.

A theorem that surprised me goes as follows. First, a classic result about *Riemannian* metrics.

**Theorem**

*If $M$ is a smooth manifold, then there is a Riemannian metric $g$ on $M$ and $(M, g)$ is a Riemannian manifold.*

This *fails* for semi-Riemannian.

**Theorem**

*There is no Lorentz metric (signature $(1, 0, 1)$) on $\mathbb{S}^2$.*

What's truly surprising is that the existence of metrics of a given signature is entirely related to the algebraic topology of the underlying manifold.

A real smooth vector bundle over a smooth manifold $M$ is an ordered pair $(E, \pi)$ where $E$ is a smooth manifold and $\pi : E \to M$ is a smooth surjection such that for all $p \in M$, $\pi^{-1}[\{p\}]$ has the structure of a finite dimensional real vector space. Moreover, for all $p \in M$ there is an open subset $\mathcal{U} \subseteq M$ with $p \in \mathcal{U}$ such that $\mathcal{U} \times \mathbb{R}^N$ is diffeomorphic to $\pi^{-1}[\mathcal{U}]$. This is an immediate generalization of the tangent bundle of a smooth manifold.

For connected vector bundles, the dimension of the vector space of the fiber of $p$ is a constant for all $p \in M$. This is the *rank* of the vector bundle.

### Theorem

*A smooth manifold $M$ has a metric with signature $(p, 0, q)$ if and only if there are smooth real vector bundles $(E, \pi_E)$ and $(F, \pi_F)$ of ranks $p$ and $q$, respectively, such that $TM \simeq E \oplus F$.*

For the case of Lorentzian manifolds, $p = N - 1$ and $q = 1$. That is, we wish to write $TM$ as the product of a rank $N - 1$ and a *line bundle*.

### Theorem

*A smooth manifold $M$ has a Lorentz metric if and only if $M$ is non-compact, or $M$ has Euler characteristic zero.*

The torus has Euler characteristic zero, and we've already given a Lorentz metric on it. There are others, such that the Lorentz metric induced by the Minkowski metric on $\mathbb{R}^2$ by the group action of $\mathbb{Z}^2$ of integer translations. The sphere has Euler characteristic 2 and is compact, so there is no Lorentz metric on it.