Homotopy Groups of Spheres

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The fundamental group is the group of equivalence classes of curves up to homotopy with the operation of concatenation. That is, given a topology space (X, τ) and a point $x_0 \in X$, we define:

$$\tilde{\pi}_1(X, x_0) = \{\, f \in C^0(\mathbb{S}^1, X) \mid f((1,0)) = x_0 \,\} \tag{1}$$

Here $C^0(\mathbb{S}^1, X)$ is the set of continuous functions from $\mathbb{S}^1$ to X. We define the equivalence relation R on $\tilde{\pi}_1(X, x_0)$ by $f \simeq g$ if and only if f is homotopic to g. The set $\pi_1(X, x_0)$ is the set of equivalence classes $\tilde{\pi}_1(X, x_0)/R$. The group operation is that of concatenating curves, it is often denoted $*$.

The higher order homotopy groups $\pi_n(X, x_0)$ are obtained by considering mappings of $\mathbb{S}^n$ into X. The group operation is obtained more easily if we think of $\mathbb{S}^n$ as the quotient space $[0,1]^n/\partial[0,1]^n$. The group operation is obtained by concatenation along the last coordinate of $[0,1]^n$.

For a path connected space (X, τ), for any two points $x_0, x_1 \in X$, the homotopy groups $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$ are isomorphic. Because of this, when considering spheres, we just write $\pi_n(\mathbb{S}^m)$ and omit the base point.

Computing the fundamental group of the circle is a well-known problem that takes up part of an algebraic topology course. The universal cover of $\mathbb{S}^1$ is the real line, and the group of deck transformations of this covering are integer shifts of the real line, $x \mapsto x + n$ for some $n \in \mathbb{Z}$. The fundamental group is isomorphic to the group of deck transformations, showing that $\pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$.

For higher order homotopy groups, the n^{th} homotopy group of a path connected, locally path-connected, semi-locally simply path connected topological space (X, τ) is isomorphic to the n^{th} homotopy group of the universal cover of (X, τ). Any continuous function $f : \mathbb{S}^n \to X$ lifts to a continuous function $\tilde{f} : \mathbb{S}^n \to \tilde{X}$, where $(\tilde{X}, \tilde{\tau})$ is the universal cover of (X, τ), since $\mathbb{S}^n$ is simply connected for $n > 1$. Any homotopy applied to $\tilde{f}$ may be projected down to a homotopy on f. Since the universal cover of $\mathbb{S}^1$ is $\mathbb{R}$, which is contractible, $\pi_n(\mathbb{S}^1)$ is trivial for all $n > 1$.

We so far have $\pi_n(\mathbb{S}^1)$ for all $n \geq 1$. Let us now compute $\pi_n(\mathbb{S}^m)$ for all $n < m$. Any continuous function $f : \mathbb{S}^n \to \mathbb{S}^m$, with $n < m$, is homotopic to a continuous function $\tilde{f} : \mathbb{S}^n \to \mathbb{S}^m$ that is *not* surjective. $\mathbb{S}^m$ minus a point is homeomorphic to $\mathbb{R}^m$ by the stereographic projection about the deleted point. Since $\mathbb{R}^m$ is contractible, this induces a homotopy between $\tilde{f}$ and a constant mapping, which shows f is homotopic to a point. Hence, $\pi_n(\mathbb{S}^m)$ is trivial for $n < m$.

Next on the list is $\pi_n(\mathbb{S}^n)$. We have computed the case when $n = 1$. Given a path connected space (X, τ) there is a homomorphism $h : \pi_n(X) \to H_n(X)$ called the *Hurewicz homomorphism*. The Hurewicz theorem says the following:

Theorem

If (X, τ) is a path connected topological space, and if $h_n : \pi_n(X) \to H_n(X)$ is the Hurewicz homomorphism, then if $n = 1$, h induces an isomorphism between $H_1(X)$ and the Abelianization of $\pi_1(X)$, and if $n > 1$ and $\pi_m(X)$ is trivial for all $m < n$, then h is an isomorphism.

We have shown that $\mathbb{S}^n$ has trivial m homotopy groups for $m < n$ meaning $\pi_n(\mathbb{S}^n)$ is isomorphic to $H_n(\mathbb{S}^n)$, which is $\mathbb{Z}$.

We now have $\pi_n(\mathbb{S}^m)$ for $n \leq m$. Homology can no longer assist us since the homology groups of a manifold are zero beyond the dimension of the manifold. If this were true of homotopy groups we'd have a rather trivial problem at hand. To compute $\pi_3(\mathbb{S}^2)$ we'll need a few concepts.

A fiber bundle is an ordered quadruple (E, B, p, F) where $p : E \to B$ is a continuous surjective function such that for all $x \in B$ the fiber $p^{-1}[\{x\}]$ is homeomorphic, with the subspace topology, to F. Moreover, p satisfies the *local trivialization* property that for all $x \in B$ there is an open subset $\mathcal{U} \subseteq B$ with $b \in \mathcal{U}$ such that $\pi^{-1}[\mathcal{U}]$ is homeomorphic to $\mathcal{U} \times F$.

A Fibration is a generalization of a fiber bundle, it is an ordered triple (E, B, p) such that for any space X and any homotopy $H : X \times [0, 1] \to B$ and for any continuous function $\tilde{f}_0$ lifting H_0, there is a homotopy $\tilde{H} : X \times [0, 1] \to E$ that lifts H. That is, $p \circ \tilde{H} = H$. If E is path connected, the fibers of any two points of B are homotopy equivalent. This homotopy equivalence class is usually referred to as *the* fiber F. Fiber bundles (E, B, p, F) are hence a special case of fibrations where the fibers of all points of B are not just homotopy equivalent, but are homeomorphic.

There is a long exact sequence of homotopy groups of a fibration. For simplicity, let us think of fiber bundles (E, B, p, F) rather than the more abstract fibrations. So the fibers of B are all homeomorphic, rather than just homotopy equivalent. There are homomorphisms:

$$\cdots \to \pi_{N+1}(F) \to \pi_{N+1}(E) \to \pi_{N+1}(B) \to \pi_N(F) \to \cdots \quad (2)$$

that make this a long exact sequence. We can compute $\pi_3(\mathbb{S}^2)$ using this sequence and the *Hopf Fibration*.

We may identify $\mathbb{S}^3$ with the unit quaternions, $\mathbf{p} \in \mathbb{H}$ with $||\mathbf{p}|| = 1$. We can then define the following continuous function $f : \mathbb{H} \to \mathbb{R}^3$:

$$f\big((a, b, c, d)\big) = \big(2(ac + bd), 2(bc - ad), a^2 + b^2 - c^2 - d^2\big) \quad (3)$$

where $(a, b, c, d) = \mathbf{p} \in \mathbb{H}$. If $||\mathbf{p}|| = 1$, then $||f(\mathbf{p})|| = 1$ showing that this is also a map from $\mathbb{S}^3$ to $\mathbb{S}^2$. With a bit of work one can see that the fiber of every point in $\mathbb{S}^2$ is a circle, showing that this is a fiber bundle $(\mathbb{S}^3, \mathbb{S}^2, f, \mathbb{S}^1)$.

The long exact homotopy sequence then yields:

$$\pi_3(\mathbb{S}^1) \to \pi_3(\mathbb{S}^3) \to \pi_3(\mathbb{S}^2) \to \pi_2(\mathbb{S}^1) \tag{4}$$

But we know $\pi_3(\mathbb{S}^1) = 0$ and $\pi_2(\mathbb{S}^1) = 0$ meaning $\pi_3(\mathbb{S}^3)$ is isomorphic to $\pi_3(\mathbb{S}^2)$. Since we've computed $\pi_3(\mathbb{S}^3)$ using the Hurewicz theorem, we have $\pi_3(\mathbb{S}^2) \simeq \mathbb{Z}$.

Equations can be boring and I'd like to visually describe $\mathbb{S}^3$ being foliated by circles. To do this we attempt to foliate $\mathbb{R}^3$ with circles first. We start with the unit circle in the xy plane. Next we place a torus of inner radius ϵ around this circle. Instead of thinking of it as a torus, we think of it as the union of planar circles each of which making the same very small angle with the xy plane. Next we consider a larger torus around our $\mathbb{S}^1$ in the xy plane, again thinking of it as the union of planar circles, but now making a larger angle with the xy plane. We continue doing this, growing the torii, thinking of it as circles making larger angles with the xy plane.

We nearly foliate all of $\mathbb{R}^3$ with circles until we reach the point where the angle between the circles and the xy plane is $\frac{\pi}{2}$. In this instance the *circles* are really the z axis, and they all lie on top of each other. We then realize that $\mathbb{S}^3$ is $\mathbb{R}^3$ with a point *at infinity* and the z axis is really a circle containing the point at infinity. This foliates the entirety of $\mathbb{S}^3$ with circles.

The following is courtesy of Niles Johnson's Sage code, released under the GNU General Public License, version 2.

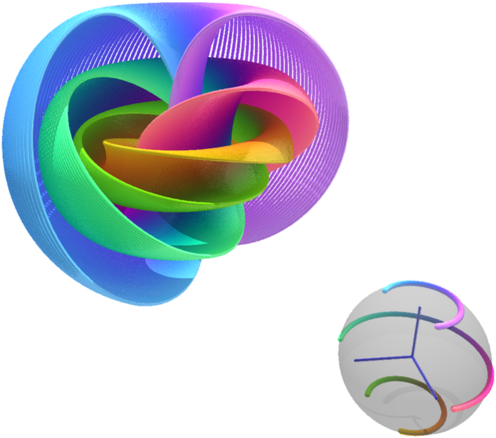

Figure: Visualization of the Hopf Fibration

The Freudenthal suspension theorem tells us the *diagonals* of the homotopy groups of spheres eventually stabilize.

Theorem

If (X, τ) is a path connected topological space with $\pi_k(X)$ trivial for all $k \leq n$ for some $n \in \mathbb{N}$, the $\pi_n(X)$ is isomorphic to $\pi_{n+1}(\Sigma X)$, where Σ is the suspension of X.

The suspension of $\mathbb{S}^n$ is $\mathbb{S}^{n+1}$ giving us the following corollary:

Theorem

$\pi_{n+k}(\mathbb{S}^n)$ is isomorphic to $\pi_{n+k+1}(\mathbb{S}^{n+1})$ for all $n > k + 1$.

Below these diagonals is all zero, as we've seen, but above these diagonals is almost random. The computations involve spectral sequences. One result of note is that $\pi_n(\mathbb{S}^m)$ is always finite except for the diagonals and super diagonals described by the previous theorem.