Curvature and the Einstein Field Equations

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Many of the notions of curvature require only a semi-Riemannian metric and a choice of affine connection.

It is almost universal amongst physicists and mathematicians to work with the unique Levi-Civita connection that a given semi-Riemannian metric induces.

We'll discuss several types of curvatures, and their uses in describing the Einstein field equations.

In physics it is common to work in a coordinate chart $(\mathcal{U}, \varphi)$ and express all physical quantities in terms of this chart. The semi-Riemannian metric g becomes a matrix $g_{\mu\nu}$ with entries $g_{\mu\nu} = g(\partial\varphi_\mu, \partial\varphi_\nu)$, which is called the *metric tensor* in general relativity. Other tensors and tensor fields will be described similarly.

The first tensor to describe is the stress-energy tensor $T_{\mu\nu}$. It is the gravitational analogue of the stress tensor from Newtonian mechanics and describes the density and flux of energy in the manifold (M, g), which is always chosen to be Lorentzian.

The Einstein field equations relate the stress-energy tensor and the metric tensor to Ricci curvature and scalar curvature.

The Ricci curvature is described in terms of the Riemann curvature tensor field (It's a tensor field, not a tensor). Given the affine connection ∇ on the semi-Riemannian manifold, the Riemann curvature tensor field is defined in one of two equivalent ways. It is a function $R : \mathfrak{X}(M)^3 \to \mathfrak{X}(M)$

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{1}$$

Where $[X, Y]$ is the Lie bracket. We can also write this as:

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \tag{2}$$

again using the Lie bracket. With this we see that the Riemann curvature tensor field measures the failure of the second derivative to commute.

If ∇ is a Levi-Civita connection (torsion free and compatible with the metric), then there are several identities the Riemann curvature tensor field enjoys. These identities can be combined with the Einstein field equations to prove the local conservation of energy and momentum, classical laws of Newtonian mechanics which still hold in general relativity.

- R is trilinear over $C^\infty(M, \mathbb{R})$.
- The Bianchi identity holds:

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0 \tag{3}$$

The Bianchi identity cyclicly permutes the vector fields. It is the Bianchi identity that helps one prove conservation of momentum and energy.

The quadruple product relates the Riemann curvature tensor field to the semi-Riemannian metric. It is defined as:

$$(X, Y, Z, T) = g\big(R(X, Y)Z, T\big) \tag{4}$$

There are several identities for this operation, which are again useful for the proof of various theorems in the framework of general relativity.

$$(X, Y, Z, T) = -(Y, X, Z, T) \tag{5}$$
$$(X, Y, Z, T) = -(X, Y, T, Z) \tag{6}$$
$$(X, Y, Z, T) = (Z, T, X, Y) \tag{7}$$

Lastly, an analogue of the Bianchi identity:

$$(X, Y, Z, T) + (Y, Z, X, T) + (Z, X, Y, T) = 0 \tag{8}$$

These identities combine to give the following theorem.

Theorem

If $(\mathcal{U}, \varphi)$ is a chart in a spacetime (M, g), if ∇ is the unique Levi-Civita connection on M, and if T is the stress-energy tensor, then:

$$\sum_{n=0}^{N-1} \nabla_{\partial\varphi_n} T_{n,m} = 0 \tag{9}$$

This is the analogue of the conservation of momentum and energy laws that occur in Newtonian mechanics. The proof is about a page and simply uses the identities of the Riemannian curvature tensor field, the quadruple product, and the Einstein field equations which will be stated soon.

The Einstein field equations relate the stress-energy tensor to the Ricci and scalar curvatures. The Ricci curvature is defined in terms of the Riemann curvature tensor field. There are two ways of doing this.

In the Riemann setting (g is positive-definite), fix $p \in M$ and $x = z_n \in T_pM$ to be unit length. Since T_pM is an n dimensional real inner product space, we may extend z_n via the Gram-Schmidt procedure to an orthonormal basis. Label these other elements $z_1, \ldots, z_{n-1}$. The Ricci curvature about p is defined as:

$$\mathrm{Ric}_p(x) = \frac{1}{n-1} \sum_{k=1}^{n} g_p\Big(R(x, z_k)x, z_k\Big) \tag{10}$$

It is a theorem that this result is independent of the choice of basis.

In the semi-Riemannian setting $T_p M$ is not an inner product space since g can, in general, fail to be positive definite. Such is the case in spacetimes with signature $(+,+,+,-)$. Fix two vector fields Y and Z. Given a vector field X, the mapping $X \mapsto R(X,Y)Z$ is linear at each tangent space. Because of this one may define the *trace* of this mapping. This is the Ricci curvature tensor.

$$\mathrm{Ric}_p(Y,Z) = \mathrm{tr}\big(X_p \mapsto R_p(X_p, Y_p)Z_p\big) \qquad (11)$$

In local coordinates $(\mathcal{U}, \varphi)$ it can be given by a matrix $R_{\mu\nu}$.

The Ricci curvature can be completely described by the sectional curvature, which is one of the older notions of curvature dating back to a time when differential geometry dealt solely with regular surfaces and curves. The sectional curvature of a 2-dimensional subspace δ of the tangent space T_pM is given by:

$$K_\delta = \frac{(v,w,v,w)}{A(v,w)} = \frac{g_p\big(R(v,w)v,w\big)}{\sqrt{||v||^2\,||w||^2 - g_p(v,w)^2}} \qquad (12)$$

where v and w are two tangent vectors that span δ, and $A(v,w)$ is the area of the parallelogram with sides v and w. K_δ is independent of choice of basis since a change of basis can be made by a combination of moves $(x,y) \mapsto (y,x)$, $(x,y) \mapsto (\lambda x, y)$, $\lambda \neq 0$, and $(x,y) \mapsto (x + \lambda y, y)$. These operations are reflection, scaling, and shearing, respectively. All of these are invariant under formula above showing K_δ is independent of basis.

For constant curvature manifolds the Ricci curvature is given by a simple formula:

$$R_{\mu\nu} = (n-1)Kg_{\mu\nu} \tag{13}$$

where K is the constant curvature of the manifold. It is probably not the case that the spacetime we live in is constant curvature.

The scalar curvature is defined directly by the Ricci curvature. Given the Riemannian definition, $\mathrm{Ric}_p(x)$, given a basis $\{z_1, \ldots, z_n\}$ of T_pM, the scalar curvature is defined by:

$$K(p) = \frac{1}{n} \sum_{k=1}^{n} \mathrm{Ric}_p(z_k) \tag{14}$$

It is independent of choice of basis. With respect to the second definition, we can define:

$$K(p) = \mathrm{tr}(R_{\mu\nu}) \tag{15}$$

The Einstein tensor is defined in terms of the Ricci and scalar tensors. We have:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} K g_{\mu\nu} \tag{16}$$

Where $R_{\mu\nu}$ is the Ricci tensor, K is the scalar curvature, and $g_{\mu\nu}$ is the metric tensor. The Einstein field equations are:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{17}$$

Where $T_{\mu\nu}$ is the stress-energy tensor. Λ is the cosmological constant, and κ is the Einstein gravitational constant.

In practice, one measures the stress-energy tensor and the Einstein tensor and wishes to solve for the metric in the Einstein field equation. A common simplification is to suppose the spacetime you are working in is a vacuum containing no mass-energy. The Einstein field equations simplify to:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{18}$$

Expanding the Einstein tensor in terms of the Ricci and scalar curvature, we get:

$$R_{\mu\nu} - \frac{1}{2} K g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{19}$$

This is a purely geometrical problem. Depending on the value of Λ there are several known spacetimes with metrics that satisfy the Einstein field equations.

- Minkowski spacetime $\mathbb{M}^{3,1}$
- Milne spacetime
- Schwarzschild vacuum spacetime
- Kerr vacuum

The value of Λ was originally thought to be zero, and Einstein retracted it from the equation. In the late 1990's it was discovered the inflation of the universe is accelerating, indicating the constant may be positive. One possible value involves the Hubble constant, given by:

$$\Lambda = 1.1056 \times 10^{-52}\,\mathrm{m}^{-2} \tag{20}$$