Numerical Experiments with the Khovanov Homology of Twist Knots

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Outline

- Conjectures on Legendrian and tranversely simple knots.
- New data for knots up to 19 crossings.
- A peculiar find about twist knots.

Legendrian Knots

A knot is a smooth embedding of $\mathbb{S}^1$ into $\mathbb{R}^3$ or $\mathbb{S}^3$. We can get a richer theory by considering the standard contact structures of these manifolds.

We seek a distribution of planes in $\mathbb{R}^3$ with the property that no surface (open or closed) can be everywhere tangent to this distribution. This is obtained in $\mathbb{R}^3$ by the one-form $\mathrm{d}z - y\mathrm{d}x$, the hyperplane at (x, y, z) being the kernel of this. This is spanned by the vectors ∂y and $y\partial z + \partial x$

Legendrian Knots

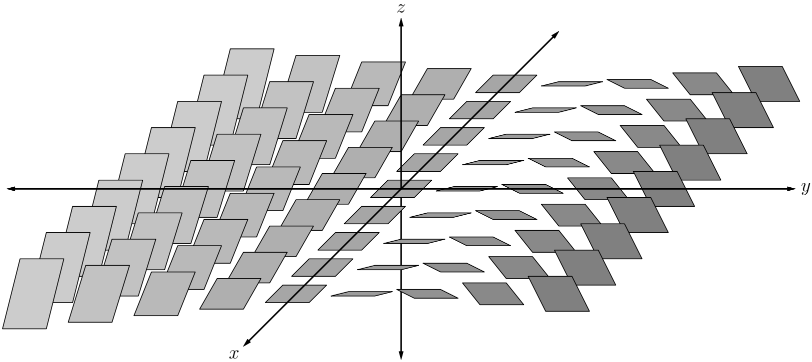

Figure: The Darboux Form for $\mathbb{R}^3$

Legendrian Knots

Contact structures can be similarly defined for odd $2k+1$ dimensional manifolds, locally given by a form α such that $\alpha \wedge (\mathrm{d}\alpha)^k \neq 0$ (This strange criterion is related to the Frobenius theorem on integrability).

Theorem (Darboux)

Given a point p in a $2k+1$ dimensional contact manifold M there is a coordinate chart $(\mathcal{U}, \varphi)$ containing p such that the contact structure is given by:

$$\alpha = d\varphi_0 - \sum_{n=1}^{k} \varphi_{2n-1} d\varphi_{2n} \tag{1}$$

So locally all contact manifolds look like the twisting hyperplane distribution given before.

Legendrian Knots

Given a $2k + 1$ dimensional contact manifold the integer k tells you the highest dimension a submanifold can be while being everywhere tangent to the contact structure. For three dimensional manifolds this is $k = 1$, so we get knots in our space.

Definition

A Legendrian submanifold in a contact manifold M is a submanifold K such that K is everywhere tangent to the hyperplane distribution. A Legendrian knot is a 1-dimensional Legendrian submanifold.

Legendrian Knots

Contact structures arise naturally from the spherical cotangent bundle of any (positive dimensional) smooth manifold M. The contact form is given by the so-called Liouville form.

Every knot can be made Legendrian by an appropriate perturbation. A *Legendrian* isotopy is an isotopy H_t of a Legendrian knot K such that H_t is legendrian for all $t \in [0, 1]$. Legendrian isotopy and topological isotopy are different notions.

Legendrian Knots

Given a (certain) spacetime (M, g) and two points, $p, q \in M$, the skies $\mathbb{S}_p$ and $\mathbb{S}_q$ of these two points are the spaces of all future-directed inextensible light-like geodesics at these two points. These live naturally in the spherical cotangent bundle of some submanifold of the spacetime so one may ask if they are linked.

Theorem (Low Conjecture)

In (certain) 2+1 dimensional spacetimes, if two points are causally related (impossible to send data between the two without exceeding the speed of light) then their skies are topologically linked.

This does not, topologically, generalize to higher dimensions.

Legendrian Knots

Theorem (Chernov, Nemirovski)

The Low conjecture holds for (certain) spacetimes if one replaces topological linking with Legendrian linking.

This hints at Legendrian linking (and knotting) having more information than topological linking.

Legendrian Knots

Legendrian equivalence implies topological equivalence since a Legendrian isotopy is indeed a topological isotopy. This does not reverse so we need invariants that distinguish Legendrian links.

By examining the Darboux form, if we have a parameterization $\gamma : [0, 1] \to \mathbb{R}^3$ of a Legendrian knot, $\gamma(t) = (x(t), y(t), z(t))$, we find the following constraint:

$$y(t) = \frac{z'(t)}{x'(t)} = \frac{\mathrm{d}z}{\mathrm{d}x} \tag{2}$$

At the left and right extremes of a knot diagram we see x' hit zero meaning the *front diagram* of the knot will have cusps.

Legendrian Knots

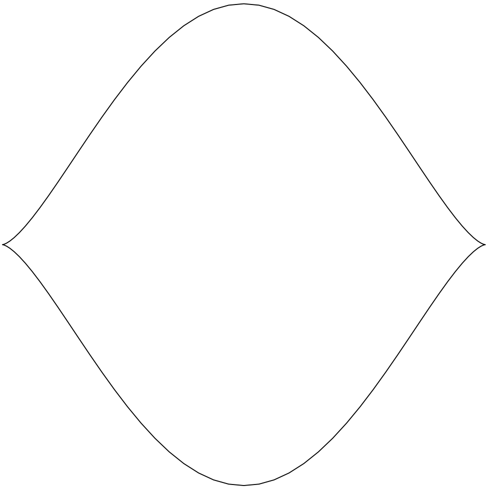

Figure: Front Diagram of a Legendrian Unknot

Legendrian Knots

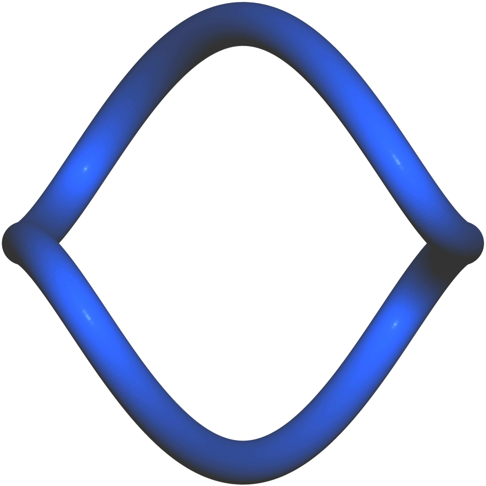

Figure: Legendrian Unknot Embedding in $\mathbb{R}^3$

Legendrian Knots

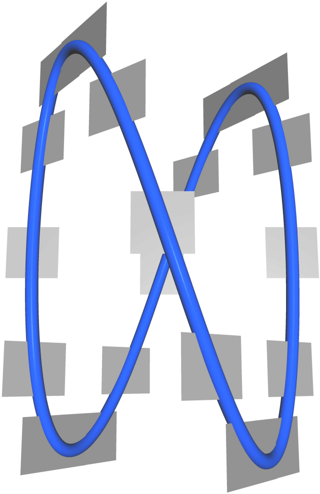

Figure: Legendrian Unknot with Hyperplane Distribution

Legendrian Knots

The Thurston-Bennequin of a front diagram is defined as the writhe minus the number of right cusps.

Definition

A Legendrian simple knot is a topological knot K such that all Legendrian representations of K are uniquely determined by their Thurston-Bennequin number tb and their rotation number rot.

There is a similar notion of *transversely simple* for transverse representations of a topological knot.

Previous Conjectures

Last year I presented the following conjecture.

If K is a topological knot (or link) type that is Legendrian (or transversally) simple, then the Khovanov homology of K distinguishes it. That is, if $\tilde{K}$ is another knot with the same Khovanov homology as K, then $\tilde{K}$ is topologically identical to K.

This is somewhat motivated by the theorem Mrowka and Kronheimer which shows that Khovanov homology is an unknot detector. Numerical evidence was presented for all knots up to 17 crossings. This has been expanded.

Previous Conjectures

It is known that the torus knots form a family of Legendrian simple knots. Etnyre, Ng, and Vertesi have also completely classified when twist knots are Legendrian simple. Indeed, the m_n twist knot is Legendrian simple if and only if $n \geq -3$. Their work further classifies when twist knots are transversally simple. Suffice it to say not all twist knots are transversally simple.

Lastly, Ng, Chongchitmate, An, Liang, and Manocha have written Java code to find Legendrian representations of knots. Their work has led to the conjecture that certain knot types are Legendrian simple, but not confirmed. For example the 6_2 knot has two distinct Legendrian representations with $(tb, rot) = (-7, 2)$ and hence this is not Legendrian simple, but the *mirror* of 6_2 may be.

Previous Conjectures

Using torus knots, twist knots, and the conjecturally Legendrian simple knots from the Legendrian knot atlas, all prime knots up to 19 crossings have been compared with these families to test the conjecture (expanding our results beyond 17).

A table of all Jones polynomials of all prime knots up to this many crossings has been tabulated (as well as Alexander and HOMFLY polynomials, but these were not directly needed for our conjecture). This data will soon be made available publically. We use this to search which knots may have the same Khovanov homology as a torus, twist, or conjecturally Legendrian simple knot (Khovanov homology categorifies the Jones polynomial).

New Finds

After analyzing the data we find several knots with the same Jones polynomial as a torus, twist, or conjecturally Legendrian simple knot. At this point we compute the Khovanov polynomial (which contains the torsion-free data of the Khovanov homology of the knots). We can make the following claims:

Theorem

If K is a prime knot with less than or equal to 19 crossings, and if T is a torus or twist knot with the same Khovanov polynomial (or Khovanov homology) as K, then T is equivalent to K.

New Finds

The degree of the Jones polynomial of a knot with n crossings is bounded by a constant multiple of n.

Using well-known formulas for the Jones polynomials of torus and twist knots we need only search through a finite set of knots to make the previous claim.

New Finds

This motivates the following question.

Question

Does the Khovanov polynomial distinguish twist knot? Does Khovanov homology?

It is known that this is true in the three simplest cases. Khovanov homology *does* distinguish the unknot, trefoils, and figure eight knot. It is also known that the Jones polynomial does *not*, the figure eight has the sames Jones polynomial as a certain 11 crossing knot.

New Finds

The strangeness here is that not all twist knots are Legendrian simple, but our search found no matches for any twist knots.

The next avenue to explore is the overall strength of the Khovanov polynomial. Perhaps the twist knots aren't too special, perhaps many knot types have unique Khovanov polynomials.

The Khovanov, Alexander, Jones, and HOMFLY polynomials for the first 350 millions knots has (nearly) been tabulated so we will soon be able to experiment with the relative strengths of these invariants.

Thank you!