Causality and Topological Linking in Two Plus One Dimensional Spacetimes

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Outline

- Spacetimes and causality.
- Cotangent bundles and spherical cotangent bundles.
- Causality in globally hyperbolic $2+1$ dimensional spacetimes.

Spacetimes and Causality

An $n+1$ dimensional spacetime is a Lorentz manifold (M, g) (a semi-Riemannian manifold with signature $(n, 1)$, or $(1, n)$ in some physics communities) with a chosen time orientation. That is, at each point a future direction is chosen and this choice is done continuously.

A time-like curve is a differentiable curve γ such that $g\big(\dot{\gamma}(t), \dot{\gamma}(t)\big) < 0$ for all t. A light-like curve is one such that $g\big(\dot{\gamma}(t), \dot{\gamma}(t)\big) = 0$ for all t. Lastly, causal curve satisfy $g\big(\dot{\gamma}(t), \dot{\gamma}(t)\big) \leq 0$.

Spacetimes and Causality

Causal curves represent the transmittance of real data since we may not exceed the speed of light. To see this, given a point $p \in M$ we may find a chart $(\mathcal{U}, \varphi)$ with $p \in \mathcal{U}$ such that:

$$g = -\mathrm{d}t^2 + \sum_{k=1}^{n} \mathrm{d}x_k^2 \tag{1}$$

where $\mathrm{d}x_k = \mathrm{d}\varphi_k$ and $\mathrm{d}t = \mathrm{d}\varphi_{n+1}$. If γ is causal, this says:

$$\sum_{k=1}^{n} \mathrm{d}x_k^2(\dot{\gamma}) \leq \mathrm{d}t^2(\dot{\gamma}) \tag{2}$$

$$\Rightarrow \sum_{k=1}^{n} \frac{\mathrm{d}x_k^2}{\mathrm{d}t^2} \leq 1 \tag{3}$$

Spacetimes and Causality

The sum of the squares of the components is the square of the norm of the velocity vector, so the square of the speed.

In natural units one takes the speed of light to be $c = 1$, so this final inequality states that the speed of the curve never exceeds that of light.

Causally related points in a spacetime (M, g) are those that can be connected by a causal curve.

Spacetimes and Causality

There are lots of spacetimes one can ponder, most of which are not physically relevant (but perhaps still fun to think about). Two reasonable restrictions are often placed on our manifolds.

- There is no time travel.
- Given two points p and q, the intersection of the causal future J_p^+ and causal past J_q^- is compact.

The set J_p^+ is the set of all points in M that can be reached by a causal future directed curve from p. Similarly J_q^- is the set of points that can be reached by causal past directed curves from q. Such a spacetime is called *globally hyperbolic*.

Spacetimes and Causality

The structure of globally hyperbolic spacetimes is well understood.

Theorem (Geroch's Splitting Theorem, 1979)

If (M, g) is a globally hyperbolic spacetime, then M is homeomorphic to $S \times \mathbb{R}$ where S is a Cauchy surface, a hypersurface such that every inextensible light-like geodesic intersects S exactly once.

This is a *topological* theorem and does not give us any smooth or geometrical information, but it is useful nonetheless. It seriously restricts the possible structure of globally hyperbolic spacetimes.

Spacetimes and Causality

A strenghthening of this theorem exists.

Theorem (Bernal, A. and Sanchez, M., 2003)

If (M, g) *is a globally hyperbolic spacetime, then there is a Riemannian Cauchy surface* S *(a Cauchy surface such that the restriction of* g *to* S *is a Riemannian metric) such that* M *is diffeomorphic to* $S \times \mathbb{R}$.

Spherical Cotangent Bundles

This talk aims to discuss the Topological Low Conjecture for $2+1$ dimensional globally hyperbolic spacetimes. Moreover, spacetimes where the Cauchy surface S has a universal cover diffeomorphic to $\mathbb{R}^2$.

To do this requires the notion of linking in the spherical cotangent bundle of a Cauchy surface. The cotangent bundle T^*M is constructed in a similar manner to the tangent bundle TM using local trivialization and smoothly *gluing* copies of $\mathbb{R}^n$ to each point in M^n. By removing the zero section we may quotient by the action of multiplication by positive real numbers and obtain the *spherical cotangent bundle* ST^*M.

Spherical Cotangent Bundles

Given a smooth manifold (no Riemannian or semi-Riemannian metric needed) there is a standard method of inventing a symplectic form on T^*M which restricts to a contact form on ST^*M. The construction is made quite explicit with the existence of a Riemannian metric g.

The metric induces a map $\tilde{g} : TM \to T^*M$ as follows. Given a vector field $X \in \mathfrak{X}(M)$ we define:

$$\tilde{g}(X)(p, v) = g(X_p, v) \tag{4}$$

This is a one-form, at each point p it takes in tangent vectors and returns real numbers, and since g is a Riemannian metric this varies smoothly. Thus $\tilde{g}$ maps vector fields to one-forms so it is a function $\tilde{g} : TM \to T^*M$.

Spherical Cotangent Bundles

Locally in some chart $(\mathcal{U}, \varphi)$ we may represent $\tilde{g}$ as a matrix with components $\tilde{g}_{i,j}$. The Liouville form Ω is then:

$$\Omega = \sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{g}_{i,j}\mathrm{d}x_i \wedge \mathrm{d}v_j + \sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial \tilde{g}_{i,j}}{\partial x_k}v_i\mathrm{d}x_j \wedge \mathrm{d}x_k \tag{5}$$

where we represent φ by $(x_1, \ldots, x_n, v_1, \ldots, v_n)$. The restriction of this to ST^*M yields a contact structure. Note since we have a Riemannian metric STM lives as a subspace of TM by considering points (p, v) with $g_p(v, v) = 1$. ST^*M similarly lives as a subspace of T^*M.

Spherical Cotangent Bundles

For the sake of visualization it helps to know the topological structure of spherical cotangent bundles. A *trivializable* tangent (or cotangent) bundle is one that is homeomorphic to $M \times \mathbb{R}^n$. Similarly a trivializable spherical tangent (or cotangent) bundle is one that may be written as $M \times \mathbb{S}^{n-1}$. Four results help.

Theorem

If the Euler characteristic of M is non-zero, then TM is not trivializable.

Theorem

If M_1 and M_2 have trivializable tangent (or cotangent) bundles, then $M_1 \times M_2$ does as well.

Theorem

Parallelizable manifolds yield trivializable bundles.

Theorem

A parallelizable manifold is orientable.

Spherical Cotangent Bundles

The only closed orientable surface with euler characteristic zero is the torus, which happens to be parallelizable. The plane is also parallelizable. For n dimensional parallelizable manifolds M the spherical cotangent bundle ST^*M is homeomorphic to $M \times S^{n-1}$.

For the torus we get $\mathbb{T}^3$, the three-torus, and for the plane we have $\mathbb{R}^2 \times \mathbb{S}^1$, the thickened torus. Both of these spaces have methods of visualizing which helps us create drawings.

Causality and Linking

Robert Low first conjectured that for causally related points p, q in (certain) $2+1$ dimensional spacetimes $(X,\ g)$ the *skies*, which live in the space of all future directed inextensible null pre-geodesics, of these points are topologically linked.

This can be made quite explicit in $2+1$ dimensional Minkowski space which is given by the semi-Riemannian metric on $\mathbb{R}^3$ g defined by:

$$g = \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}t^2 \qquad (6)$$

The future direction at each point is *up*, i.e. the positive t direction (which coincides with the z axis in $\mathbb{R}^3$).

Causality and Linking

Given two points $p, q \in \mathbb{M}^{2,1}$ that are causally related either there exists a light ray between them or not. If there is the skies intersect, which is non-trivial. Otherwise the skies form two nested circles in the Cauchy surface $\mathbb{R}^2$ (the larger circle is the one for the event further in the *past*).

To make life simpler we can suppose p and q have the same spacial component and only differ in time. That is, $p = (x, y, t_0)$ and $q = (x, y, t_1)$ with $t_0 < t_1$. In this case the two skies are actually concentric circles in $\mathbb{R}^2$ centered at (x, y).

Causality and Linking

The skies embed naturally into $ST^*\mathbb{R}^2$, which is diffeomorphic to the thickened torus as follows. The plane is homeomorphic to the open unit disk via:

$$f(x, y) = \frac{(x, y)}{1 + \sqrt{x^2 + y^2}} \tag{7}$$

We may parameterize the thickened torus by elements of the unit disk $\mathbb{D}^2$ and points on the circle $\mathbb{S}^1$. Given a curve in the plane we use f to map it to the disk, and then examine angles given by the unit normal to the curve. This gives us a point in the disk and an angle on the circle, yielding a unique point in the thickened torus.

Causality and Linking

For two concentric circles we end up with a parameterization of the Hopf link. This is shown below.

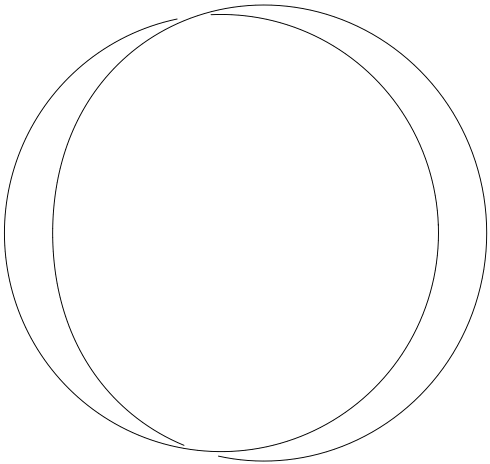

Figure: Linking Detects Causality

Causality and Linking

It is possible to get other links. For two distinct points in the same Cauchy surface (which are hence not causally related) it is also easy to show that the skies form unlinked disjoint circles.

If one considers the Riemannian metric induced by pullback of the mapping $F : \mathbb{R}^2 \to \mathbb{R}$ by:

$$F(x, y) = \frac{1}{1 + x^2 + y^2} \tag{8}$$

(This is a surface in $\mathbb{R}^3$ so we may steal the standard metric) and warp product this with $\mathbb{R}$ one gets a Minkowski-like space with a *bump* around the origin. Causally related points can yield different topological links, such as the Whitehead link, in this scenario.

Causality and Linking

The topological Low conjecture does hold for $2+1$ dimensional globally hyperbolic spaces where the Cauchy surface admits a covering by an open domain in $\mathbb{R}^2$. In $3+1$ and higher dimensions this is false, Low himself found a counterexample. Here one must replace *topological* linking with *Legendrian* linking, where we consider the Liouville form on the spherical cotangent bundle of the Cauchy surface.

This becomes hard to visualize, even for the Minkowski space $M^{3,1}$ since the spherical cotangent bundle is a five dimensional topological manifold. For $M^{3,1}$ it is $\mathbb{R}^3 \times \mathbb{S}^2$.

Causality and Linking

Theorem (Chernov, Nemirovski 2008)

If (X, g) is a globally hyperbolic spacetime, if M is a spacelike (Riemannian) Cauchy surface of dimension $m \geq 2$, and if M has a smooth covering by an open subset of $\mathbb{R}^m$, then two causally related points $p, q \in X$ have Legendrian linked skies in ST^*M.

This hints to us that Legendrian knot and link theory may be much richer than its topological counterpart.

Thanks!