Morse Theory and Borde-Sorkin Spacetimes

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Outline

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- Borde-Sorkin Spacetimes
- The Borde-Sorkin Conjecture

Motivation

Since the late 90s some physicists and mathematicians have been exploring spacetimes with *broken* parts, isolated points with degenerate metrics, and changing spacial topologies.

Many have also rejected this since they often have bad topological properties. Borde and Sorkin proposed a resolution to this issue by considering topology-changing spacetimes that are constructed via *Morse functions*.

These *Borde-Sorkin* spacetimes may be necessary in the development of a consistent theory of quantum gravity. Borde and Sorkin have conjectured that these spacetimes are *causally continuous*, so long as the index of the Morse points is not 1 or $n-1$.

Morse Theory

Marston Morse (C.E. 1892 - 1977) observed that much of the topology of a smooth manifold M can be observed from a general smooth function $f : M \to \mathbb{R}$. For smooth surfaces S embedded in $\mathbb{R}^3$ the *elevation* function serves as a good example.

Given $(x, y, z) \in S$ define $f(x, y, z) = z$. The fiber of a real number r yields *contour lines* on the surface. You may also find isolated points and **X**'s (double points). Higher order points are possible, like triple points where the contour lines meet like an asterisk *****, but small perturbations in f will remove this.

Morse Theory

If $\mathbf{p} = (x,\, y,\, z) \in S$ is not a critical point, meaning the gradient $\nabla(f)_{\mathbf{p}}$ is non-zero, and if $r = f(\mathbf{p})$, then it seems intuitive, at least by experiment, that $f^{-1}\big[(-\infty,\, r)\big]$ and $f^{-1}\big[(-\infty,\, r + \varepsilon)\big]$ are homeomorphic for small enough ε. That is, the topology does not change until one reaches critical points. The elevation function on the standard embedding of $\mathbb{T}^2$ into $\mathbb{R}^3$ serves as the quintessential example.

Morse Theory

The *index* of a non-degenerate (non-singular Hessian matrix) critial point $\mathbf{p}$ is the maximum number of independent directions in which the value of f *decreases* from $\mathbf{p}$.

We may define this for more general manifolds by saying the index is the maximum value possible for the dimension of a subspace of $T_{\mathbf{p}}S$ such that the Hessian is negative definite ($vHv^T \leq 0$ for all $v \in T_{\mathbf{p}}S$ and $vHv^T = 0$ if and only if $v = \mathbf{0}$). By Sylvester's Law of Inertia this value is independent of chart chosen for $\mathbf{p}$.

Morse Theory

A Morse function on a smooth manifold M is a smooth function $f : M \to \mathbb{R}$ that has no degenerate critical points.

Theorem

If M is a smooth manifold, then there exists a Morse function $f : M \to \mathbb{R}$.

Theorem

If M is a smooth manifold, then most functions $f \in C^\infty(M, \mathbb{R})$ are Morse functions. That is, the set of Morse functions is open and dense in the C^2 topology on $C^\infty(M, \mathbb{R})$.

Borde-Sorkin Spacetimes

Spacetimes are Lorentz manifolds (signature $(n, 1)$) with a chosen time orientation. At no points in the spacetime is the metric allowed to deviate from being non-degenerate, nor may the signature change. Sorkin proposed the following construction in 1989.

Let $\mathcal{M}$ be a compact cobordism of dimension n (a smooth manifold with boundary such that $\partial\mathcal{M}$ is the disjoint union of two closed $n-1$ dimensional smooth manifolds). Let h be a Riemannian metric on $\mathcal{M}$, $\zeta > 1$ a constant, and $f : \mathcal{M} \to \mathbb{R}$ a Morse function (the notation here follows Garcia-Heveling, 2022).

Borde-Sorkin Spacetimes

By the Morse lemma, at a given critical point p of f there is a chart $(\mathcal{U}, \varphi)$ such that:

$$f(x) = \sum_{k=1}^{n} a_k \varphi_k(x)^2 \tag{1}$$

for all $x \in \mathcal{U}$ where $a_k \neq 0$ for each k. It follows that critical points of the Morse function are topologically isolated.

Borde-Sorkin Spacetimes

The *Morse Metric* of $(\mathcal{M}, h, f, \zeta)$ is defined on $\mathcal{M} \times \mathbb{R}$ via:

$$g = ||\mathrm{d}f||_h^2 h - \zeta \mathrm{d}f^2 \tag{2}$$

We let $M = \mathrm{Int}(\mathcal{M})$, and abuse notation by endowing M with the subspace metric $g|_M$ and labelling this g. At non-critical points of f we have a spacetime. At the critical points $\mathrm{d}f$ is zero, we see that $g = 0 - 0 = 0$, a degenerate metric.

Borde-Sorkin Spacetimes

Note that, for non-critical points $p \in M$, (M, g) is somewhat *locally* globally hyperbolic (locally globally is a strange phrase). That is, locally (M, g) has the structure of a Riemann manifold crossed with $\mathbb{R}$, M acting locally as a Cauchy surface.

Borde-Sorkin spacetimes can intuitively be thought of as globally hyperbolic spacetimes *glued* together at the critical points of the Morse function.

Borde-Sorkin Conjecture

These spacetimes were invented to allow topology-changing without the poor properties such features usually exhibit. It has been conjectured that the pathologies arise with casually discontinuous spaces.

Following Hawking and Sachs, 1973, casually continuous spacetimes are, roughly speaking, those in which the future and past of an observer vary in a well behaved manner under continuous perturbations to the metric g or to the position of the observer.

Conjecture (Borde-Sorkin)

The Morse spacetime (M, g) *is casually continuous if and only if the index of all critical points of* f *is different than* 1 *and* $n - 1$.

Borde-Sorkin Conjecture

Theorem (Garcia-Heveling, 2022)

If (M, g) is a Borde-Sorkin spacetime coming from a Morse geometry $(\mathcal{M}, h, f, \zeta)$, if f has one critical point p_c, and if there is a chart $(\mathcal{U}, \varphi)$ containing p_c such that:

$$f(x) = \sum_{k=1}^{n} a_k \varphi_k(x)^2 \tag{3}$$

$$h = \sum_{k=1}^{n} d\varphi_k^2 \tag{4}$$

where $a_k \neq 0$ for all k, and where:

$$\frac{1}{\zeta} < \left|\frac{a_k}{a_j}\right| < \zeta \quad and \quad \frac{5}{8} \leq \left|\frac{a_k}{a_j}\right| \leq \frac{8}{5} \tag{5}$$

For all a_k, a_j, then (M, g) is causally continuous.