Legendre Polynomials and Saturn

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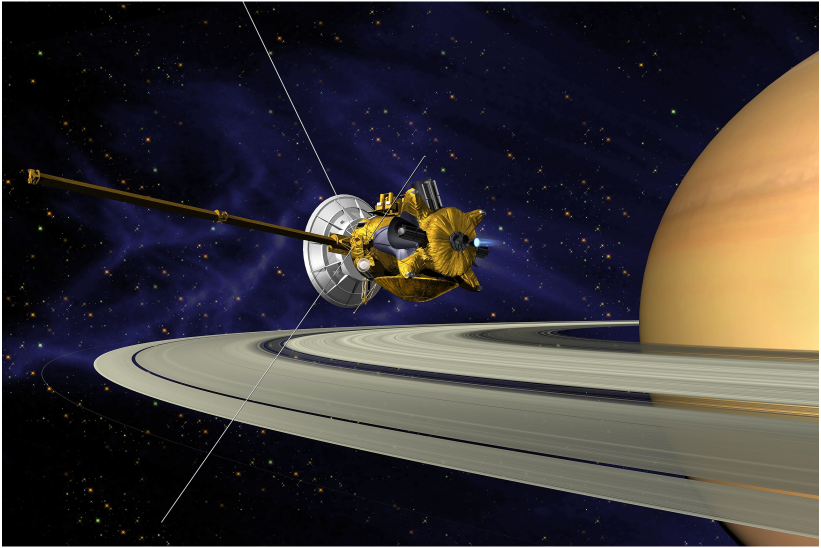
The Cassini Mission



The Cassini Mission

The Cassini orbiter was a space probe sent to Saturn back in the 90s. Launched in 1997, it took about 7 years to get there, arriving in 2004 (the distance to Saturn is about 1 billion miles).

Before burning up in the atmosphere of Saturn, Cassini spent 13 years orbiting the planet collecting data on the rings, poles, and more. In this talk we'll discuss some of the details of the radio science mission.

The Cassini Mission

As Cassini orbited Saturn, occasionally its rings got between the the probe and Earth. This phenomenon is called an *occultation*. Radio waves sent to Earth are then diffracted by the rings, and one observes a diffraction pattern on Earth. The goal is to reconstruct the ring profile from this diffracted pattern using Fourier optics.

Fourier Optics and the Fresnel Transform

The rings of Saturn are very circular, so the *optical transmittance* can be approximated by $T(\rho, \phi) = T(\rho)$, where ϕ is the azimuth angle in the ring plane, and ρ is the radial distance from a point in the plane to the core of Saturn. Fourier optics tells us that the diffracted profile can be computed via:

$$\hat{T}(\rho_0) = \int_0^{\infty} \int_0^{2\pi} \rho\, T(\rho) \frac{e^{i\psi(\rho, \rho_0, \phi, \phi_0)}}{D} \mathrm{d}\rho\, \mathrm{d}\phi \qquad (1)$$

where D is the distance from the point (ρ, ϕ) to the observer (the Cassini probe), and ψ is the *Fresnel kernel*.

Fourier Optics and the Fresnel Transform

This double integral is quite hard to work with, but can be approximated by a single integral using the *stationary phase approximation*. We observe for an integral of the form:

$$I = \int_a^b e^{i\phi(t)} \mathrm{d}t \tag{2}$$

where ϕ grows faster-than-linear, most of the contribution comes from where ϕ is roughly stationary. That is, as ϕ grows faster and faster, $e^{i\phi(t)}$ starts to oscillate rapidy, meaning the regions under the curve start to cancel each other out and contribute little to the integral.

Fourier Optics and the Fresnel Transform

By applying this to the Fresnel transform, we get:

$$\hat{T}(\rho_0) = K \int_0^\infty T(\rho) e^{i\psi(\rho,\phi_s,\rho_0,\phi_0)} \mathrm{d}\rho \tag{3}$$

where ϕ_s is the *stationary value* of ψ, where $\partial\psi/\partial\phi$ is zero, and K is (roughly) some constant. $\hat{T}$ is the measured quantity, and T is the transmittance of the rings, the value we wish to solve for. These *integral equations* generally yield very hard inverse problems. We try some approximations.

Fourier Optics and the Fresnel Transform

The Fresnel kernel has the form:

$$\psi = kD\Big(\sqrt{1 - 2\xi + \eta} + \xi - 1\Big) \tag{4}$$

where:

$$\xi = \frac{\cos(B)}{D}\big(\rho\cos(\phi) - \rho_0\cos(\phi_0)\big) \tag{5}$$

$$\eta = \frac{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0)}{D^2} \tag{6}$$

Where B is the angle made with the ring plane and the line from the observer (Cassini) to the detector (Earth).

The Fresnel Approximation

A decent approximation to the stationary value ϕ_s is $\phi_s = \phi_0$. We can improve this by applying one iteration of Newton's method to the Fresnel kernel ψ. This yields the *Fresnel approximation*. The constant and linear term for ψ ends up being zero, and by truncating the power series for ψ to the quadratic we get:

$$\hat{T}(\rho_0) = K\int_0^\infty T(\rho)e^{i\frac{\pi}{2}\left(\frac{\rho-\rho_0}{F}\right)^2}\mathrm{d}\rho \tag{7}$$

where F is the so-called *Fresnel scale*. The constant K can be expressed in terms of the Fresnel scale as well, yielding:

$$\hat{T}(\rho_0) = \frac{1-i}{F}\int_0^\infty T(\rho)e^{i\frac{\pi}{2}\left(\frac{\rho-\rho_0}{F}\right)^2}\mathrm{d}\rho \tag{8}$$

The Fresnel Approximation

As long as F can be treated as constant (a usually safe assumption), this final equation is a *convolution* of T with the simplified Fresnel kernel. The convolution theorem, which states that the Fourier transform of a convolution is the product of the individual Fourier transforms, then tells us how to invert this equation for T. We get:

$$T(\rho) = \frac{1+i}{F} \int_0^\infty \hat{T}(\rho_0) e^{-i\frac{\pi}{2}\left(\frac{\rho-\rho_0}{F}\right)^2} \mathrm{d}\rho_0 \tag{9}$$

Note the similarity of this inversion with the Fourier transform. The Fresnel transform and the Fourier transform are very closely related.

The Fresnel Approximation

For certain geometrical scenarios the approximation does not hold and one has to perform something like Newton's method or Halley's method to get an accurate value for the stationary phase approximation.

This is slow. We use Legendre polynomials to speed up this calculation.

Legendre Polynomials

Legendre polynomials are defined as an *orthogonal* system of polynomials on the interval $[-1, 1]$. That is, $P_n(x)$ is a sequence of polynomials satisfying:

$$\int_{-1}^{1} P_n(x)P_m(x) = 0 \tag{10}$$

whenever $n \neq m$. The Sturm-Liouville equation that defines them is:

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0 \tag{11}$$

Legendre Polynomials

The generating function of the Legendre polynomials is:

$$\sum_{n=0}^{\infty} P_n(x)t^n = \frac{1}{\sqrt{1-2xt+t^2}} \tag{12}$$

We're seen something like this already. Recall the formula for the Fresnel kernel:

$$\psi = kD\Big(\sqrt{1-2\xi+\eta}+\xi-1\Big) \tag{13}$$

The partial derivative of ψ with respect to ϕ, evaluated at $\phi = \phi_0$, can then be represented via the derivative of the generating function. In particular, the stationary value can be efficiently approximated by Legendre polynomials.

Legendre Polynomials

By using quartic or octic expansions, we can roughly the same speed as the quadratic Fresnel approximation, but still be able to apply the approximation in more extreme geometries (such as low B values). The speed improvements over the full Newton-method computation is about 100x or more, depending on the data set.