Linking Numbers and Causality in Spacetimes

Knots in Washington $\frac{\texttt{\sim 0xF4}}{\texttt{1 << 5}}$

Ryan Maguire

December 10, 2023

Outline

- Spacetimes
- Causality and Linking
- Linking and Affine Linking Numbers
- Gravitational Lensing

Spacetimes

A Riemannian manifold is a smooth manifold M equipped with a *Riemannian metric*, a means of measuring angles between tangent vectors that varies smoothly between tangent spaces. That is, for each $p \in M$ there is a positive-definite symmetric bilinear form $g_p : T_pM \times T_pM \to \mathbb{R}$ and for every pair of smooth vector fields X, Y the function

$$p \mapsto g_p(X_p, \ Y_p) \tag{1}$$

is smooth.

Semi-Riemannian manifolds (also called pseudo-Riemannian manifolds) generalize this slightly. The positive-definite condition is relaxed to non-degeneracy. For each $p \in M$ and for all non-zero $v \in T_pM$ there is some tangent vector $w \in T_pM$ such that $g_p(v, \ w) \neq 0$.

Spacetimes

Sylvester's theorem of inertia tells us that for every $p \in M$ there is a chart $(x_1, \cdots, x_n)$ and three numbers n_0, n_+, n_- that are constant (independent of the chart) such that $n = n_0 + n_+ + n_-$ and:

$$g_p = \sum_{k=1}^{n_+} \mathrm{d}x_k^2 - \sum_{k=1}^{n_-} \mathrm{d}x_{k+n_+}^2 \tag{2}$$

Non-degeneracy means $n_0 = 0$. The *signature* of a semi-Riemannian metric is the ordered pair (n_+, n_-).[1]

[1] Riemannian manifolds have signature $(n, 0)$.

Spacetimes

A *Lorentz* manifold is a semi-Riemannian manifold (X, g) of dimension $n+1$ with signature $(n, 1)$. That is, for every point $p \in X$ there is a chart $(x_1, \cdots, x_n, t)$ such that:

$$g_p = -\mathrm{d}t^2 + \sum_{k=1}^{n} \mathrm{d}x_k^2 \tag{3}$$

The *null vectors* in T_pM are those that satisfy $g_p(v, v) = 0$. The equation above tells us that these vectors satisfy the equation of a cone.

Spacetimes

Intuitively, null vectors represent particles traveling at the speed of light. We can see this by abusing notation and letting $c = 1$. We have:

$$-c^2 \mathrm{d}t^2 + \sum_{k=1}^{n} \mathrm{d}x_k^2 = 0 \tag{4}$$

$$\Rightarrow \sum_{k=1}^{n} \left(\frac{\mathrm{d}x_k}{\mathrm{d}t}\right)^2 = c^2 \tag{5}$$

$$\Rightarrow \sqrt{\sum_{k=1}^{n} \left(\frac{\mathrm{d}x_k}{\mathrm{d}t}\right)^2} = c \tag{6}$$

This last equation says the *speed* the tangent vector represents is equal to c, which we're taking to be the speed of light.

Spacetimes

The *light-cone* at p, the set of all null vectors in T_pM, splits T_pM into three parts: the interior, the exterior, and the cone itself. The interior consists of two connected components.

A *time-orientation* is a smooth choice of one of these connected components as one varies the point p.[2] The chosen component is called the *future direction* at the point p.

A spacetime is a Lorentz manifold (X, g) with a time orientation.

---

[2]Note: orientable and time-orientable are different. The (open) Möbius strip is not orientable, but it can be time-orientable, pending the metric g. An annulus is orientable, but it can be non-time-orientable, again pending g. All four combinations of orientable and time-orientable are possible. These are separate notions.

Spacetimes

The future and past of a point p, denoted J_p^+ and J_p^-, respectively, are all points in X that can be connected to p by a curve γ whose speed is always bounded by the speed of light. That is:

$$g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)) \leq 0 \tag{7}$$

for all t.

Nice spacetimes have the following properties:

- $J_p^+ \cap J_q^-$ is compact for all $p, q \in X$.
- No time travel.

No time travel means there is no future-pointing curve γ (not exceeding the speed of light) that starts at a point p and later returns to p.

These spacetimes are called *globally hyperbolic*.

Spacetimes

Theorem (Geroch, 1970)

Globally hyperbolic spacetimes (X, g) *are homeomorphic to* $M \times \mathbb{R}$ *for some topological manifold* M.

Theorem (Bernal-Sánchez, 2003)

Globally hyperbolic spacetimes (X, g) *are diffeomorphic to* $M \times \mathbb{R}$ *for some smooth manifold* M*, and the restriction of* g *to* $M \times \{t\}$ *can be made Riemannian for all* $t \in \mathbb{R}$.

Moreover, $M \times \{t\}$ is a *Cauchy hypersurface*, a submanifold with the property that particles traveling at less-than-or-equal-to the speed of light will intersect M at precisely one time moment t_0.

Causality and Linking

The *sky* of a point p in a spacetime (X, g) is the space of all light-rays passing through p.

If (X, g) is globally hyperbolic, the sky of a point p can be given a more explicit topological and geometric structure. Let M be a Cauchy surface so that X is diffeomorphic to $M \times \mathbb{R}$ and let $t_0 \in \mathbb{R}$ be the unique time such that $p \in M \times \{t_0\}$.

Since $(M, g|_M)$ is Riemannian the spherical cotangent bundle ST^*M can be realized as unit-length elements of T^*M.

For any ray of light γ passing through p the linear functional $\phi : T_pM \to \mathbb{R}$ defined by:

$$\phi(v) = g_p\big(v, \dot{\gamma}(t_0)\big) \tag{8}$$

is non-zero. It hence defines a point in ST^*M.

Causality and Linking

The sky of a point can thus be identified with the fibre of p under the canonical projection $\mathrm{proj}_M : ST^*M \to M$.

The skies of two points $p, q \in X$ can be used to answer questions about *causality*. That is, whether or not it is possible for p and q to communicate with each other by sending messages that do not travel faster than light.

In some sense the skies of p and q may be *linked* in ST^*M.

Causality and Linking

ST^*M naturally has a contact structure given by the Liouville form. One may then ask of the skies of two points are *Legendrian linked*, instead of just topologically linked. This leads to the following theorem.

Theorem (Chernov, Nemirovksi, 2009)

The skies of causally related points in a globally hyperbolic spacetime are Legendrian linked.

For the remainder of the talk we'll talk about other linking ideas that can be used to study causality.

Linking and Affine Linking Numbers

The linking number of a link with two components can be obtained by allowing a component of the link to pass through itself, but not the other component. For any two component link you will eventually obtain two loops that wrap around each other.

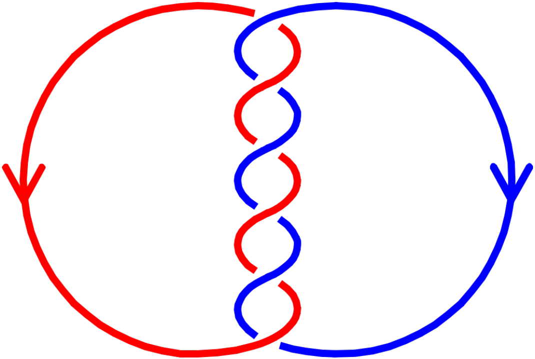


Figure: Linking-Number Three

Linking and Affine Linking Numbers

We generalize this to higher dimensional spheres in the spherical cotangent bundle of a manifold M via the the *affine linking number*.

We say that the two spheres $\mathbb{S}_p$ and $\mathbb{S}_q$ that are the fibers of two points $p, q \in M$ under the canonical projection $\mathrm{proj}_M : ST^*M \to M$ have affine linking number zero.

Given two arbitrary spheres in ST^*M we perturb them until we obtain two new spheres that are just the fibers of points p and q. By keeping track of the number of times a double point occurs during this perturbation (and the sign of the double point), we get the *affine linking number*.

Linking and Affine Linking Numbers

For most manifolds, it does not matter how you undergo this perturbation.

Theorem (Chernov, Rudyak, 2008)

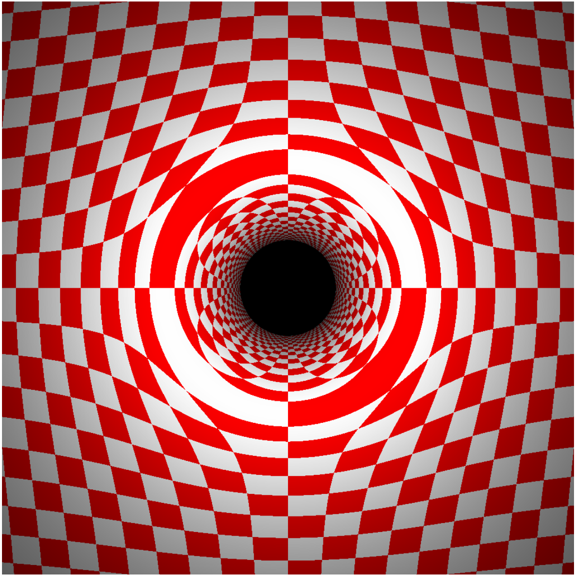
If M is not an odd-dimensional rational homology sphere, then the affine linking number is an invariant.

Gravitational Lensing

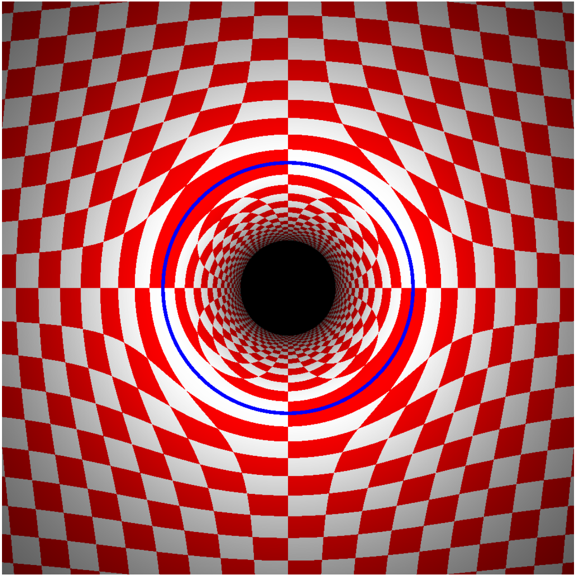
If the curvature of a spacetime is extreme enough, light from one point can be seen in various directions and it is possible to see the same point multiple times (even infinitely many).

This phenomena is known as gravitational lensing. It has been observed and well documented by astronomers.

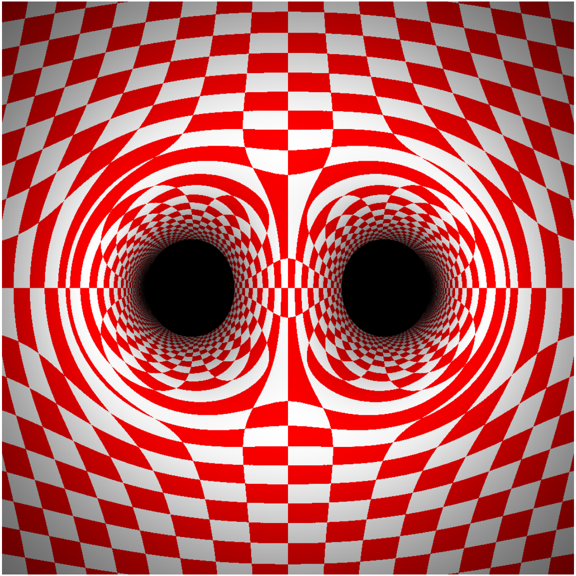
Gravitational Lensing



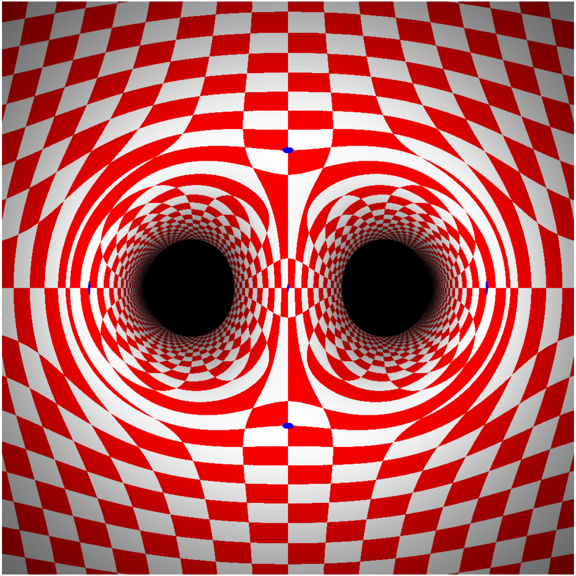
Gravitational Lensing



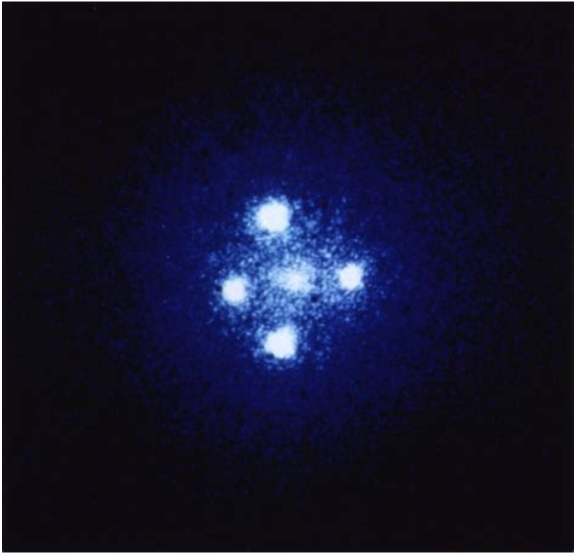
Gravitational Lensing



Gravitational Lensing



Gravitational Lensing



Gravitational Lensing

Affine linking number can be used to detect how many times an observer sees a given point.

Theorem (Chernov, Maguire, 2023)

Assume (X, g) is globally hyperbolic and the Cauchy surface of it is not an odd dimensional rational homology sphere. Let $\gamma : (-\varepsilon, \varepsilon) \to X$ be a small timelike curve that passes through p at time moment 0. If:

$$alk(\mathbb{S}_{\gamma(\varepsilon)}, \mathbb{S}_{q}) - alk(\mathbb{S}_{\gamma(-\varepsilon)}, \mathbb{S}_{q}) = N \tag{9}$$

for all small ε, then the observer at p sees light from q coming from at least N different directions.

If the time-like sectional curvatures of X are non-negative, this becomes sharp. The observer at p sees *exactly* N copies of the point q.

Thank You!