Khovanov Homology and Legendrian Simple Knots

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Outline

1. Knots and invariants.
2. Contact topology.
3. Results.

Knots and Links

Intuitively we all know what a knot is, a loop of string in space that may be tangled in some way. They were a part of various cultures long before they became of interest to the mathematical community.

1. The Celtic book of Kells has many complicated knot drawings.
2. The Viking legend of Hildr contains the Borromean rings.
3. Tibetan Buddhism uses the endless knot as a religious symbol.
4. The hammer of Thor, Mjölnir, occasionally depicts the Whitehead link.

Other knots can be found in Islamic, Jewish, and Japanese cultures, with some examples being well over 1000 years old.

Knots and Links

Mathematically knots are relatively new. Early investigations occurred in the 1700s and early 1800s, but the theory picked up popularity in the mid and late 1800s for two reasons.

1. Many problems in electromagnetism involved charged loops of wire that may be linked together.
2. Vortex theory, an early attempt at explaining the structure of atoms, used knots and links to describe chemical properties.

Vortex theory was eventually disproved by J. J. Thompson, ironically as he was attempting to provide evidence for it. Knots were not entirely abandoned and were absorbed into topology in the early 20^{th} century.

Knots and Links

Why might we care about knots and links? Two theorems I like.

Theorem (Lickorish-Wallace, 1960)

Every compact orientable connected three dimensional manifold can be obtained by taking the 3-sphere $\mathbb{S}^3$ *and cutting out a link L and gluing it back together with some twists.*[1]

Theorem (Steinitz-Tait, 1877, 1922)

Every connected planar graph with signed edges can be represented by a knot diagram. Conversely, every knot diagram can be represented by two connected graphs with signed edges that are related to each other as planar duals.

[1] In technical terms, this is a Dehn surgery.

Knots and Links

So what is a knot? A **knot** is a smooth or polygonal embedding $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$. A **knot diagram** is a projection of a knot onto the plane where we mark the overlapping points in some way so that we can easily identify which part is going over and which is under.

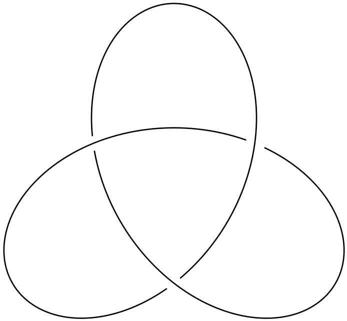

Figure: Knot Diagram

We'll say two knots are **equivalent** if we can smoothly deform one into the other without tearing or cutting.

Knot diagrams turn geometric and analytical objects (functions and embeddings) into combinatorial ones. In the 1920s it was shown that the notion of knot equivalence can be demonstrated using these diagrams and the three **Reidemeister moves**.

Knots and Links

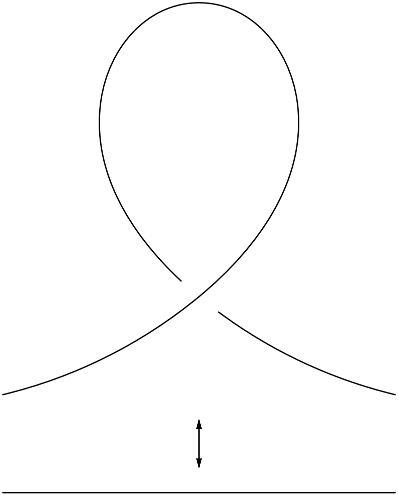

Figure: Type I Move

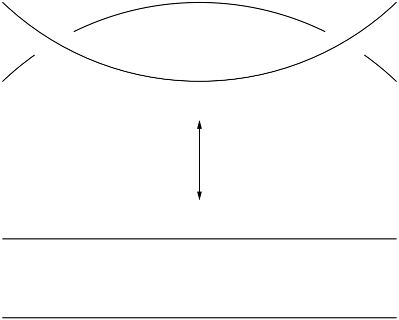

Figure: Type II Move

Knots and Links

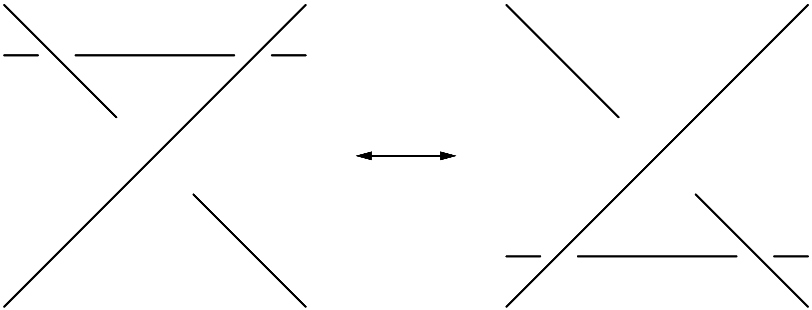

Figure: Type III Move

Knots and Links

In practice we now have a means of distinguishing two knot diagrams. The number of moves can be quite high, however. Even if one of the diagrams is a simple circle with no other crossings, if the other diagram has n crossing it may take more than $10^{26} \times n^{11}$ Reidemeister moves to undo it.[2]

Determining if a knot diagram represents an unknotted circle then takes roughly $3^{10^{26} \times n^{11}}$ steps and we quickly approach the *transcomputational problem* as n increases.[3]

Instead of doing this we invent *invariants*, which are usually algebraic objects attached to knot diagrams that do not change with the three Reidemeister moves. We'll be working with two of these.

[2] This is the best known bound at the time of this writing.

[3] See the *Bremermann limit* for details on transcomputation.

The Jones Polynomial

The Jones polynomial assigns a Laurent polynomial, a polynomial in q and q^{-1} with integer coefficients, to knot diagrams and is an invariant. We can define it pictorially using the Kauffman bracket in terms of smoothings.

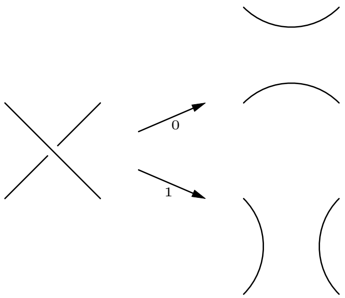

Figure: Smoothing Crossings

The Jones Polynomial

The Kauffman bracket is defined recursively. Given a knot (or link) diagram L we write:

$$\langle \emptyset \rangle = 1 \tag{1}$$

$$\langle \mathbb{S}^1 \sqcup L \rangle = (q + q^{-1})\langle L \rangle \tag{2}$$

$$\langle L \rangle = \langle L_{n,0} \rangle - q\langle L_{n,1} \rangle \tag{3}$$

where $\mathbb{S}^1 \sqcup L$ represents the disjoint union of L and a circle. The Jones polynomial $J_L(q)$ is obtained from $\langle L \rangle$ by normalizing it by a factor $\pm q^k$ for a particular k that depends on the diagram. The computation of this factor is linear in the number of crossings so we can restrict our attention to the Kauffman bracket.

The Jones Polynomial

If we follow this recursive formula we end up with 2^n different ways of completely smoothing the diagram so that there are no crossings left.

If we label the crossings 1 to n then every possible smoothing uniquely corresponds to a number between 1 and 2^n. Write your number $1 \leq k \leq 2^n$ in binary. If the m^{th} bit is 0 do a 0-smoothing at the m^{th} crossing, otherwise do a 1-smoothing. The resulting 2^n pictures is called the **cube of resolutions** for the diagram.

The Jones Polynomial

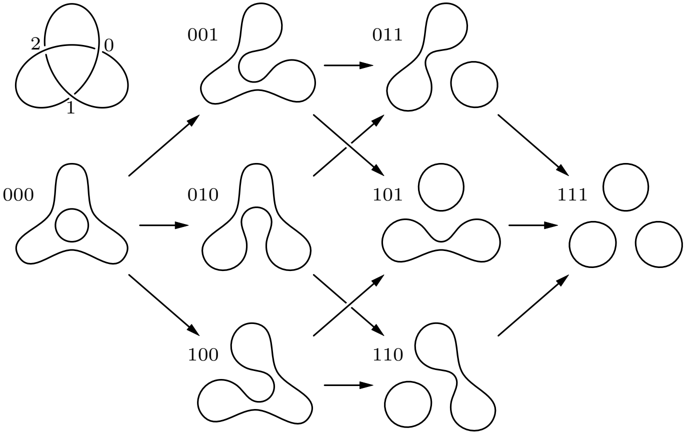

Figure: Cube of Resolutions for the Trefoil

The Jones Polynomial

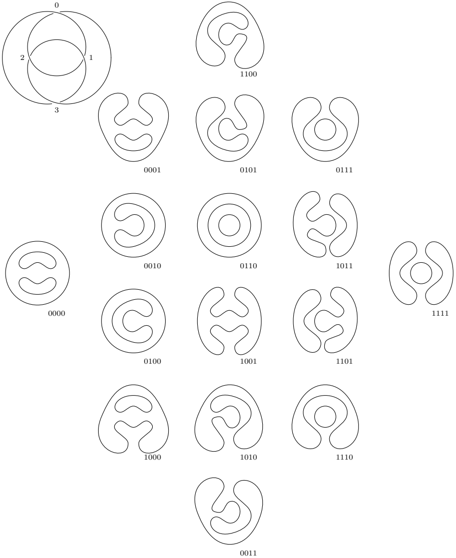

Figure: Cube of Resolutions for the Figure-Eight

The Jones Polynomial

The Kauffman bracket is obtained by counting the cycles in the smoothings and summing over them with appropriates weights. We obtain the formula

$$\langle L \rangle = \sum_{k=1}^{2^n} (-q)^{w(k)} (q + q^{-1})^{c(k)} \tag{4}$$

where $w(k)$ is the Hamming weight, the number of 1's that occur in the binary expansion of k, and $c(k)$ is the number of cycles corresponding to the k^{th} complete smoothing.

The Jones Polynomial

Several algorithms exist for the computation of the Jones polynomial and Kauffman bracket, and the complexity is known to be **NP-Hard**. The computation has been studied in the quantum aspect and a polynomial-time additive approximation method exists.

The algorithm we'll discuss is one of the simplest, but it lends itself to a clever speed-up trick and a means of computing Khovanov homology as well, which we'll soon discuss.

We first define *PD* code. Take a knot diagram and pick a starting point. Trace your finger around the knot and label the arcs in increasing order. At each under crossing write `X[a,b,c,d]` where `a` is arc you're walking on, `b` is the strand to your right, `c` is the strand in front, and `d` is the strand to the left.

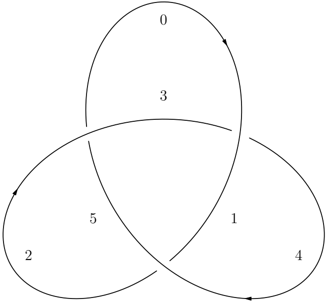

Figure: Trefoil with Arcs Labeled

The Jones Polynomial

The *PD* code of the diagram is the ordered sequence of quadruples `X[a,b,c,d]` for each under crossing. For the trefoil on the previous page we get `X[1,5,2,4]`, `X[3,1,4,0]`, `X[5,3,0,2]`.

The Kauffman bracket is now computed with a *symbolic calculus*.

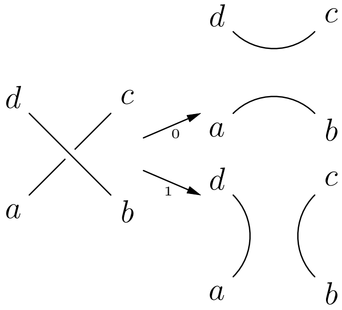

Figure: Motive for the Symbolic Calculus

The Jones Polynomial

The Kauffman relation tells us to replace `X[a,b,c,d]` with `P[a,b]P[c,d]` $- q$`P[a,d]P[b,c]`. The Kauffman bracket becomes a product of these formal polynomials:

$$\prod_{k=1}^{n} \left(\mathtt{P[ak,bk]P[ck,dk]} - q\mathtt{P[ak,dk]P[bk,cd]}\right) \qquad (5)$$

After expanding we have formal products of ordered pairs `P[a,b]`. We simplify via `P[a,b]P[b,c]` $\mapsto$ `P[a,c]` and `P[a,b]P[a,b]` $\mapsto$ `P[a,b]`. The Kauffman relation then tells us to remove cycles and replace them with $q + q^{-1}$. This amounts to `P[a,a]` $\mapsto q + q^{-1}$.

The Jones Polynomial

This gives us a quick algorithm to implement, but we can improve it. Rather than smoothing all of the crossings at once and expanding we pick one and resolve it using

$$\texttt{X[a,b,c,d]} \mapsto \texttt{P[a,b]P[c,d]} - q\texttt{P[a,d]P[b,c]}$$

We then choose the next crossing `X[e,f,g,h]` with the most numbers in common with `X[a,b,c,d]`. We perform the same replacement and simplify. We then add the next crossing with the most numbers in common with the ones we've already worked with.

By adding crossings in this way we grow our *computational front* minimally, ensuring we have less work to do in the end. This simple trick can dramatically improve performance.

The Jones Polynomial

We've implemented several algorithms for the Jones polynomial. Using this we were able to tabulate the invariant for all prime knots up to 19 crossings, over 352 million knots.

We also examined the strength of the invariant. Below is the legend for the table on the next page.

Keyword	Description
Cr	Crossing number, largest number of crossings considered.
Unique	Number of polynomials that occur for one knot.
Almost	Number of polynomials that occur for two knots.
Total	Total number of distinct polynomials in list.
Knots	Total number of knots in list.
FracU	Unique / Total
FracT	Total / Knots
FracK	Unique / Knots

Table: Legend for the Statistics Table

The Jones Polynomial

Cr	Unique	Almost	Total	Knots	FracU	FracT	FracK
03	1	0	1	1	1.000000	1.000000	1.000000
04	2	0	2	2	1.000000	1.000000	1.000000
05	4	0	4	4	1.000000	1.000000	1.000000
06	7	0	7	7	1.000000	1.000000	1.000000
07	14	0	14	14	1.000000	1.000000	1.000000
08	35	0	35	35	1.000000	1.000000	1.000000
09	84	0	84	84	1.000000	1.000000	1.000000
10	223	13	236	249	0.944915	0.947791	0.895582
11	626	77	710	801	0.881690	0.886392	0.781523
12	1981	345	2420	2977	0.818595	0.812899	0.665435
13	6855	1695	9287	12965	0.738129	0.716313	0.528731
14	25271	7439	37578	59937	0.672495	0.626958	0.421626
15	105246	35371	170363	313230	0.617775	0.543891	0.336002
16	487774	173677	829284	1701935	0.588187	0.487260	0.286600
17	2498968	894450	4342890	9755328	0.575416	0.445181	0.256164
18	13817237	4863074	24116048	58021794	0.572948	0.415638	0.238139
19	82712788	27409120	141439472	352152252	0.584793	0.401643	0.234878

Table: Statistics for the Jones Polynomial

Khovanov Homology

Khovanov homology is our next invariant, which generalizes the Jones polynomial. We replace a polynomial with a homological object. The Khovanov bracket is defined via

$$[[\emptyset]] = 0 \to \mathbb{Z} \to 0 \tag{6}$$
$$[[\mathbb{S}^1 \sqcup L]] = V \otimes [[L]] \tag{7}$$
$$[[L]] = \mathcal{F}\big(0 \to [[L_{n,0}]] \to [[L_{n,1}]]\{1\} \to 0\big) \tag{8}$$

where V is a graded vector space of graded dimension $q + q^{-1}$ [4] and $\mathcal{F}$ is the flatten operation that takes a double complex into a chain complex by direct sums along diagonals.

[4] Free modules work too.

Khovanov Homology

The differential between $[[L_{n,0}]]$ and $[[L_{n,1}]]\{1\}$ is defined pictorially. We use the cube of resolutions of the knot or link diagram, for example. Recalling our previous binary notation, if two strings differ in only one place then there is an edge between them in the cube of resolutions.

The edge describes a cobordism (a pair of pants) that either fuses two cycles into one or splits a cycle into two. Fusing amounts to a homomorphism between $V \otimes V$ and V, whereas splitting needs a map from V to $V \otimes V$.

Khovanov Homology

These are the homomorphisms.

$$m(v_- \otimes v_-) = \mathbf{0} \tag{9}$$
$$m(v_- \otimes v_+) = v_- \tag{10}$$
$$m(v_+ \otimes v_-) = v_- \tag{11}$$
$$m(v_+ \otimes v_+) = v_+ \tag{12}$$
$$\Delta(v_-) = v_- \otimes v_- \tag{13}$$
$$\Delta(v_+) = v_+ \otimes v_- + v_- \otimes v_+ \tag{14}$$

The differential is defined by an alternating sum of these homomorphisms along strings of equal hamming weight.

The Jones Polynomial

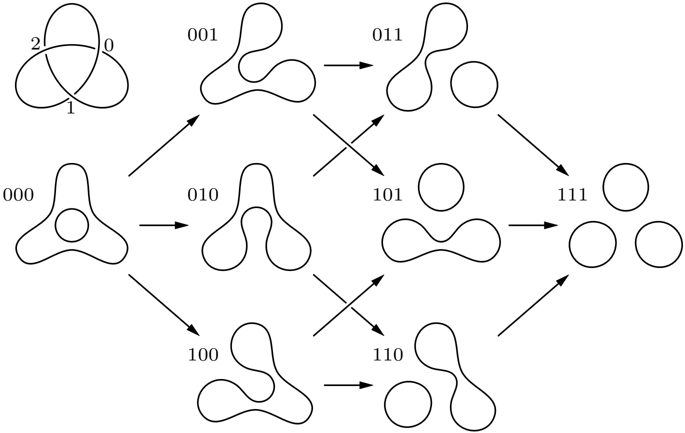

Figure: Cube of Resolutions for the Trefoil

Khovanov Homology

The differential does indeed square to zero and we get a homology out of this. The r^{th} homology group $Kh^r(L)$ is the direct sum of homogeneous parts $Kh^r_s(L)$ and the Khovanov polynomial of the diagram is the Poincaré polynomial of the homology

$$Kh_L(q,\, t) = \sum_{r,\, s} q^r t^s \dim\big(Kh^r_s(L)\big) \tag{15}$$

The Jones polynomial is recovered via

$$J_L(q) = Kh_L(q,\, -1) \tag{16}$$

Khovanov Homology

The symbolic calculus can be modified for Khovanov homology and the Khovanov polynomial.

By experimenting with the `JavaKh` library we were able to tabulate the Khovanov polynomial of all prime knots up to 17 crossings. This took about two months, 19 crossings would have taken two years. We're currently trying to speed up the computations to make this more approachable.

Khovanov Homology

Cr	Unique	Almost	Total	Knots	FracU	FracT	FracK
03	1	0	1	1	1.000000	1.000000	1.000000
04	2	0	2	2	1.000000	1.000000	1.000000
05	4	0	4	4	1.000000	1.000000	1.000000
06	7	0	7	7	1.000000	1.000000	1.000000
07	14	0	14	14	1.000000	1.000000	1.000000
08	35	0	35	35	1.000000	1.000000	1.000000
09	84	0	84	84	1.000000	1.000000	1.000000
10	225	12	237	249	0.949367	0.951807	0.903614
11	641	71	718	801	0.892758	0.896380	0.800250
12	2051	326	2462	2977	0.833063	0.827007	0.688949
13	7223	1636	9539	12965	0.757207	0.735750	0.557115
14	27317	7441	39222	59937	0.696471	0.654387	0.455762
15	118534	36867	182598	313230	0.649153	0.582952	0.378425
16	578928	187639	919835	1701935	0.629382	0.540464	0.340159
17	3167028	1001101	5033403	9755328	0.629202	0.515965	0.324646

Table: Statistics for the Khovanov Polynomial

HOMFLY-PT

Slight digression, the HOMFLY-PT polynomial was also investigated. It too generalizes the Jones polynomial (and also the Alexander polynomial).

It is different than the Khovanov polynomial, there are knots with different HOMFLY-PT polynomials but the same Khovanov polynomial, and vice-versa.

Using the `regina` library we've tabulated the HOMFLY-PT polynomial of all prime knots up to 19 crossings.

HOMFLY-PT

Cr	Unique	Almost	Total	Knots	FracU	FracT	FracK
03	1	0	1	1	1.000000	1.000000	1.000000
04	2	0	2	2	1.000000	1.000000	1.000000
05	4	0	4	4	1.000000	1.000000	1.000000
06	7	0	7	7	1.000000	1.000000	1.000000
07	14	0	14	14	1.000000	1.000000	1.000000
08	35	0	35	35	1.000000	1.000000	1.000000
09	84	0	84	84	1.000000	1.000000	1.000000
10	241	4	245	249	0.983673	0.983936	0.967871
11	730	34	765	801	0.954248	0.955056	0.911361
12	2494	210	2724	2977	0.915565	0.915015	0.837756
13	9475	1302	11044	12965	0.857932	0.851832	0.730814
14	39401	7170	48329	59937	0.815266	0.806330	0.657374
15	186799	38833	238614	313230	0.782850	0.761785	0.596364
16	979987	209669	1266261	1701935	0.773922	0.744013	0.575808
17	5559808	1157938	7175287	9755328	0.774855	0.735525	0.569925
18	33722920	6480965	42857755	58021794	0.786857	0.738649	0.581211
19	213355372	36387952	264839694	352152252	0.805602	0.752060	0.605861

Table: Statistics for the HOMFLY-PT Polynomial

Khovanov Homology

Back to Khovanov homology. While it has been conjectured that the Jones polynomial distinguishes the unknot, it is *known* that Khovanov homology does.

Theorem (Kronheimer-Mrowka, 2011)

If a knot has the same Khovanov homology as the unknot, then it is equivalent to the unknot.

It is now known the Khovanov homology also detects the trefoils, figure eight, and the cinquefoils. These results will motivate our work discussed later.

Contact Topology

Before diving into the results, lets discuss contact topology. The theory derives itself from physics, classical Hamiltonian mechanics in particular. Particles in n dimensions can be described by $2n$ coordinates, their position and momentum. This is the *phase space* coordinates.

By considering hypersurfaces of constant kinetic energy we obtain $2n - 1$ dimensional objects. The properties of these manifolds are axiomatize to create contact structures.

Contact Topology

A **contact manifold** is a smooth $2n+1$ dimensional manifold X together with a collection of smooth charts $(\mathcal{U}_i, \varphi_i)$ and one-forms α_i such that the charts cover the manifold and the α_i satisfy

$$\alpha_i \wedge (\mathrm{d}\alpha_i)^n = 0 \tag{17}$$

and such that α_i and α_k agree whenever $\mathcal{U}_i$ and $\mathcal{U}_k$ overlap.

Contact Topology

The kernels of the one-forms describe co-dimension one planes in the tangent space of each point in the manifold. This strange condition on the α is called *maximal non-integrability*. It means there is no hypersurface of dimension greater than *n* that is everywhere tangent to this collection of planes.

The Darboux theorem tells us locally any such manifold has a chart $(\mathcal{U}, \varphi)$ where the one-form α is given by:

$$\alpha = \sum_{k=1}^{n} \mathrm{d}\varphi_{2k} - \varphi_1 \mathrm{d}\varphi_{2k+1} \tag{18}$$

Contact Topology

For $\mathbb{R}^3$ this tells us we get a contact structure by using a single global chart and the one-form

$$\alpha = \mathrm{d}z - y\mathrm{d}x \tag{19}$$

This is the standard contact structure on $\mathbb{R}^3$. At each point (x, y, z) we see that ∂y and $\partial x + y\partial z$ span the kernel of α meaning we can explicitly draw the hyperplane distribution.

Contact Topology

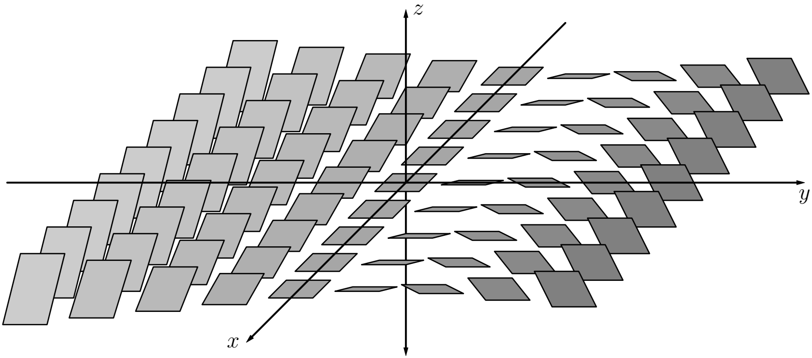

Figure: Standard Contact Structure on $\mathbb{R}^3$

Contact Topology

While it is impossible for a surface to be everywhere tangent, it is possible for curves, or *knots*, to be. A *Legendrian* knot is a smooth embedding $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ such that $\alpha(\dot{\gamma}(t)) = 0$ for each $t \in \mathbb{S}^1$.

This restriction takes away a degree of freedom from the knot since the y coordinate must satisfy

$$\mathrm{d}z - y\mathrm{d}x = 0 \tag{20}$$

$$\Rightarrow y = \frac{\mathrm{d}z}{\mathrm{d}x} \tag{21}$$

$$\Rightarrow y(t) = \frac{\mathrm{d}z/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} \tag{22}$$

$$\Rightarrow y(t) = \frac{\dot{z}(t)}{\dot{x}(t)} \tag{23}$$

Contact Topology

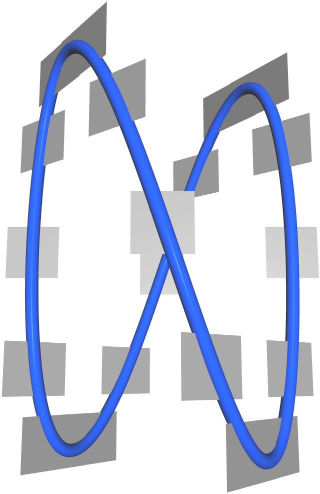

Figure: Legendrian Unknot

Contact Topology

For this to be well defined when $\dot{x}(t) = 0$ we also need $\dot{z}(t)$ to approach zero as well. The value y is also finite, and since the circle is compact the range of y is also bounded. Hence in a knot diagram there will be no *vertical tangencies* and instead we obtain cusps.

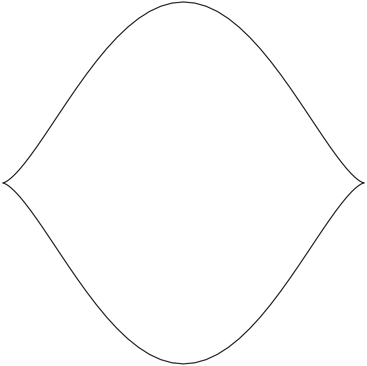

Figure: Legendrian Unknot Diagram

Contact Topology

Two Legendrian knots are **equivalent** if we can smoothly deform one into the other, keeping the knot Legendrian at each stage of the deformation.

It is possible for two knots to be topologically equivalent but different as Legendrian embeddings. To distinguish Legendrian knots then requires Legendrian invariants. The two simplest are the Thurston-Bennequin *tb* and rotation numbers *rot*. A **Legendrian simple** knot is a knot where all Legendrian embeddings are uniquely determined by these two invariants.

It is known that are torus knots are Legendrian simple.

Contact Topology

The contact structure also allows us to describe transverse knots, those that are everywhere transverse to the distribution of hyperplanes. We can also define transverse invariants and transversally simple knots.

The twist knot knots with a positive number of twist serve as our example of transversally simple knots.

Results and Conjectures

The knots where Khovanov homology is known to uniquely distinguish are all Legendrian simple. We've conjectured that all such knots may be detectable.

We computed the Jones polynomial of all prime knots of up to 19 crossings and compared these with the Jones polynomial of torus knots. At the end of this computation four matches were found.

Results and Conjectures

Torus Knot	Non-Torus Knot	Jones Polynomial
$T(2,5)$	dciaFHjEbg	$-q^{14}+q^{12}-q^{10}+q^{8}+q^{4}$
$T(2,7)$	fJGkHlICEABd	$-q^{20}+q^{18}-q^{16}+q^{14}-q^{12}+q^{10}+q^{6}$
$T(2,11)$	gHlImJnKBDFAce	$-q^{32}+q^{30}-q^{28}+q^{26}-q^{24}+q^{22}-q^{20}+q^{18}-q^{16}+q^{14}+q^{10}$
$T(2,5)$	iNHlPJqCoKFmdABgE	$-q^{14}+q^{12}-q^{10}+q^{8}+q^{4}$

Table: Knots whose Jones polynomial matches that of a Torus Knot

From this the unknot conjecture cannot be generalized to torus knots or Legendrian simple knots. In each case the Khovanov polynomials are different.

Theorem

If a prime knot K has less than or equal to 19 crossings and has the same Khovanov polynomial, or Khovanov homology, as a torus knot, than it is equivalent to it.

Results and Conjectures

A similar search was performed with the twist knots for transversally simple knots. A lot more matches were found but in each case the Khovanov polynomials differed.

Twist Knot	Non-Twist Knot	Jones Polynomial
m_2	eikGbHJCaFd	$q^4 - q^2 + 1 - q^{-2} + q^{-4}$
m_3	dgikFHaEjbc	$-q^{12} + q^{10} - q^8 + 2q^6 - q^4 + q^2$
m_3	gfJKHlaIEBCD	$-q^{12} + q^{10} - q^8 + 2q^6 - q^4 + q^2$
m_3	hGJaMlCdEKBfI	$-q^{12} + q^{10} - q^8 + 2q^6 - q^4 + q^2$
m_5	bhDGijCkaef	$-q^{16} + q^{14} - q^{12} + 2q^{10} - 2q^8 + 2q^6 - q^4 + q^2$
m_6	cefIgbajkDh	$q^{12} - q^{10} + q^8 - 2q^6 + 2q^4 - 2q^2 + 2 - q^{-2} + q^{-4}$
m_6	femIbaJKLCGHd	$q^{12} - q^{10} + q^8 - 2q^6 + 2q^4 - 2q^2 + 2 - q^{-2} + q^{-4}$
m_6	jpIFNMrClqOhkEDabg	$q^{12} - q^{10} + q^8 - 2q^6 + 2q^4 - 2q^2 + 2 - q^{-2} + q^{-4}$
m_7	cgjFHIaDEkb	$-q^{20} + q^{18} - q^{16} + 2q^{14} - 2q^{12} + 2q^{10} - 2q^8 + 2q^6 - q^4 + q^2$
m_8	knIHoBjCDQrMPaeLgF	$q^{16} - q^{14} + q^{12} - 2q^{10} + 2q^8 - 2q^6 + 2q^4 - 2q^2 + 2 - q^{-2} + q^{-4}$
m_9	jopIFMrDlqNhkEabcg	$-q^{24} + q^{22} - q^{20} + 2q^{18} - 2q^{16} + 2q^{14} - 2q^{12} + 2q^{10} - 2q^8 + 2q^6 - q^4 + q^2$

Table: Knots whose Jones polynomial matches that of a Twist Knot

Results and Conjectures

Theorem

If a prime knot K has 19 or fewer crossings and the same Khovanov homology or Khovanov polynomial as a twist knot, then it is equivalent to it.

An interesting thing to note is that not all twist knots are transversally or Legendrian simple. This may lead one to conjecture that Khovanov homology is able to detect twist knots in general.

Results and Conjectures

We also looked through the conjectured Legendrian simple knots in the Legendrian knot atlas. Once again many matches were found for the Jones polynomial, but the Khovanov polynomials were all different.

Ng Knot	Matching Knot	Jones Polynomial
$m(6_2)$	glfoJcbKMNDaHIe	$q^{10} - 2q^8 + 2q^6 - 2q^4 + 2q^2 - 1 + q^{-2}$
$m(6_2)$	hknEGmDbJLaIfc	$q^{10} - 2q^8 + 2q^6 - 2q^4 + 2q^2 - 1 + q^{-2}$
$m(6_2)$	gKHlmIdJCEABf	$q^{10} - 2q^8 + 2q^6 - 2q^4 + 2q^2 - 1 + q^{-2}$
$m(6_2)$	ehkjmGIaFlcbd	$q^{10} - 2q^8 + 2q^6 - 2q^4 + 2q^2 - 1 + q^{-2}$
$m(7_3)$	hgelkIbaJFcd	$-q^{18} + q^{16} - 2q^{14} + 3q^{12} - 2q^{10} + 2q^8 - q^6 + q^4$
$m(7_4)$	gfkHlbjIDaec	$-q^{16} + q^{14} - 2q^{12} + 3q^{10} - 2q^8 + 3q^6 - 2q^4 + q^2$
$m(9_{48})$	gnoqKDjIMrpEaHblfc	$q^2 - 3 + 4q^{-2} - 4q^{-4} + 6q^{-6} - 4q^{-8} + 3q^{-10} - 2q^{-12}$
$m(9_{49})$	lFKJIOAEnDCpBhmG	$q^{-4} - 2q^{-6} + 4q^{-8} - 4q^{-10} + 5q^{-12} - 4q^{-14} + 3q^{-16} - 2q^{-18}$
$m(10_{128})$	eHPNqGJlBFoiaDCkM	$-q^{20} + q^{18} - 2q^{16} + 2q^{14} - q^{12} + 2q^{10} - q^8 + q^6$
$m(10_{128})$	edjkaGIlFbch	$-q^{20} + q^{18} - 2q^{16} + 2q^{14} - q^{12} + 2q^{10} - q^8 + q^6$
$m(10_{136})$	igDKHJaEbFC	$q^6 - 2q^4 + 2q^2 - 2 + 3q^{-2} - 2q^{-4} + 2q^{-6} - q^{-8}$
10_{145}	eoHKqGJnCFmPDibaL	$-q^{20} + q^{18} - q^{16} + q^{14} + q^4$
10_{145}	kNJIpHLFECoMGABd	$-q^{20} + q^{18} - q^{16} + q^{14} + q^4$
10_{161}	hOqrljsnMeipFAgkbcd	$-q^{22} + q^{20} - q^{18} + q^{16} - q^{14} + q^{12} + q^6$

Table: Conjectured Legendrian Simple Knots

Future Work

We were able to implement several algorithms and get computations for the Jones, HOMFLY-PT, and Alexander (not discussed here) polynomials in a reasonable amount of time. All three invariants have been tabulated to prime knots up to 19 crossings.

The Khovanov computation was still too slow. The previous tabulation effort stopped at 16, and we've been able to push this to 17. By introducing parallel computing and making some optimizations we may be able to get to 19 crossings in a few months, instead of a few years.

The End

Thank You!