# All Horses are the Same Color

Ryan Maguire

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To prove infinitely many statements simultaneously we often use *mathematical induction*. If we have statements $P_0$, $P_1$, $P_2$, and so on, and we know that $P_0$ is true, and also that $P_n$ implies $P_{n+1}$, then every statement is true. Let's *prove* that all horses are the same color. We will prove this by induction. The $n^{th}$ statement is:

Given a collection of $n$ horses, every horse is the same color.

The base case says that if we have 1 horse, then the horse is one color. This is true (if the horse is multicolored, we'll say the color is multicolored).

Having proved the base case, let's prove the inductive step. We suppose the statement is true for $n$ horses, and have to prove this implies the statement is true for $n+1$ horses. So suppose that if we have $n$ horses, then all horses are the same color. Let us prove that $n+1$ horses must all have the same color. Number the horses 1, 2, 3, $\cdots$, $n$, $n+1$. First, group the horses together 1 to $n$. Since this group of horses has size $n$, by the induction hypothesis, each horse must be the same color. Next, group the horses 2 to $n+1$. Since this group of horses has size $n$, each horse must be the same color. But since these two groups overlap, the two colors must be the same. Therefore, all $n+1$ horses are the same. By induction, we have proved that all horses are the same color.

This is obviously absurd and false, but *why*? What is wrong with the proof? Work together with your group to figure it out.