Domains of Functions - Example 1

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A result known back to the times of Euclid (300 B.C.E) states that if a and b are real numbers, and if $ab = 0$, then either $a = 0$, or $b = 0$ (or both). The proof is quite simple. If a is not zero, then we can divide both sides of $ab = 0$ by a, and we obtain $b = 0$. Similarly, if b is not zero, then we can divide both sides of $ab = 0$ by b and obtain $a = 0$. Hence, if $ab = 0$, then at least one of these must be zero. We can use this result to find for which real numbers a formula may be undefined.

Let's consider the following expression:

$$f(x) = \frac{1}{x(1-x^2)} \tag{1}$$

For what subset of the real numbers is this formula well-defined? The only problem we could encounter is a division-by-zero. The denominator of the expression is $x(1-x^2)$, so we need to exclude real numbers where this evaluates to zero. Using the statement from the first paragraph, if $x(1-x^2) = 0$, then either $x = 0$, or $1 - x^2 = 0$. The expression $1 - x^2 = 0$ has two solutions, 1 and -1. We can see this by factoring, obtaining $1 - x^2 = (1+x)(1-x)$. So, in total, there are 3 real numbers where this formula is not well-defined: 0, 1, and -1. Using the notation from set theory, we can write the domain of f via:

$$D = (-\infty, -1) \cup (-1, 0) \cup (0, 1) \cup (1, \infty) \tag{2}$$

The function is plotted in Fig 1.

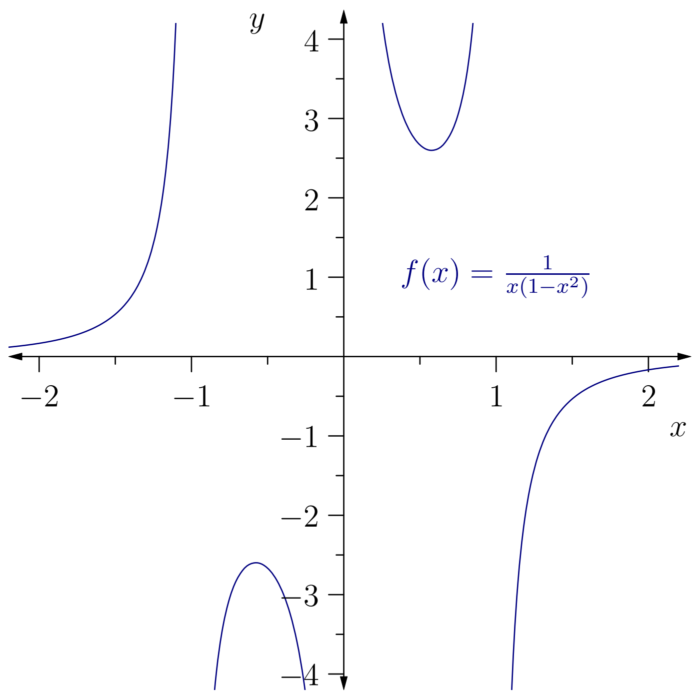

Figure 1: Graph of the function f