Domains of Functions - Example 2

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When an expression involves logarithmic functions such as the natural log we need to be careful that the input is positive. The log base b (with $b > 1$) of a real number x returns the number y such that $x = b^y$. Since b^y can never be negative, $\log_b(x)$ has no meaning for negative values of x. For 0, we can say $\log_b(0) = -\infty$ since $b^{-\infty} = 0$ (more precisely, b^x tends to zero as x approaches $-\infty$), but $-\infty$ is not a real number and so 0 must be excluded from the input of $\log_b$. A plot of 2^x is given in Fig. 1 indicating that 2^x is never negative and tends to 0 as x tends to $-\infty$.

Consider the expression below:

$$f(x) = \frac{1}{x\ln(x)} \tag{1}$$

There are a few restrictions on x for $f(x)$ to be well-defined. Firstly, we have a division so we need the denominator to be non-zero. We must avoid $x\ln(x) = 0$. If $x\ln(x) = 0$, then either $x = 0$ or $\ln(x) = 0$. For any $b > 1$, $\log_b(x) = 0$ is true precisely when $x = 1$. If we want $1 = b^y$, we use that fact that for any non-zero real number b it is true that $b^0 = 1$. This gives us $\log_b(1) = 0$. So, in particular, $\ln(1) = 0$. To avoid a division-by-zero in our expression we need $x \neq 0$ and $x \neq 1$. There is another restriction. We need that $\ln(x)$ is well-defined as well. This occurs when $x > 0$. So, in total we need $x \neq 0$, $x \neq 1$, and $x > 0$. The requirement $x > 0$ excludes 0 so we can rid ourselves of the first requirement, and need $x > 0$ and $x \neq 1$. We can write the domain of f as:

$$D = (0, 1) \cup (1, \infty) \tag{2}$$

The function is plotted in Fig. 2.

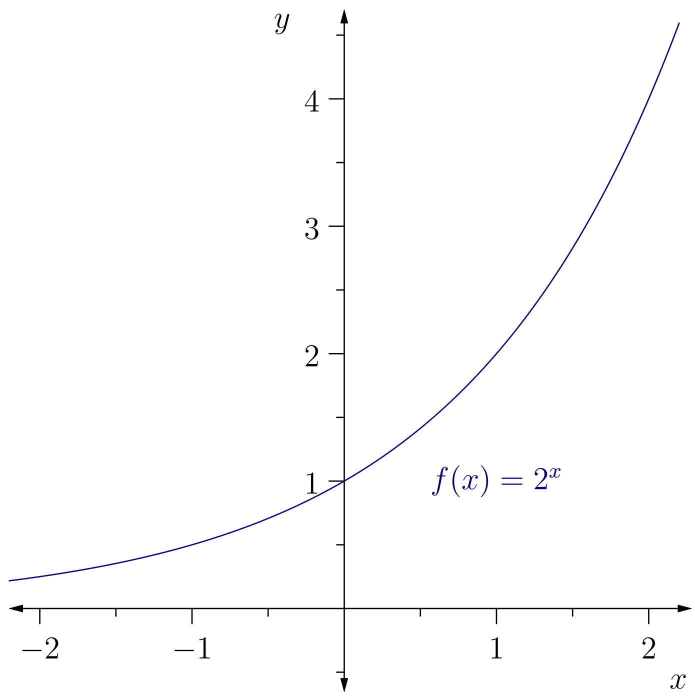

Figure 1: The function 2^x

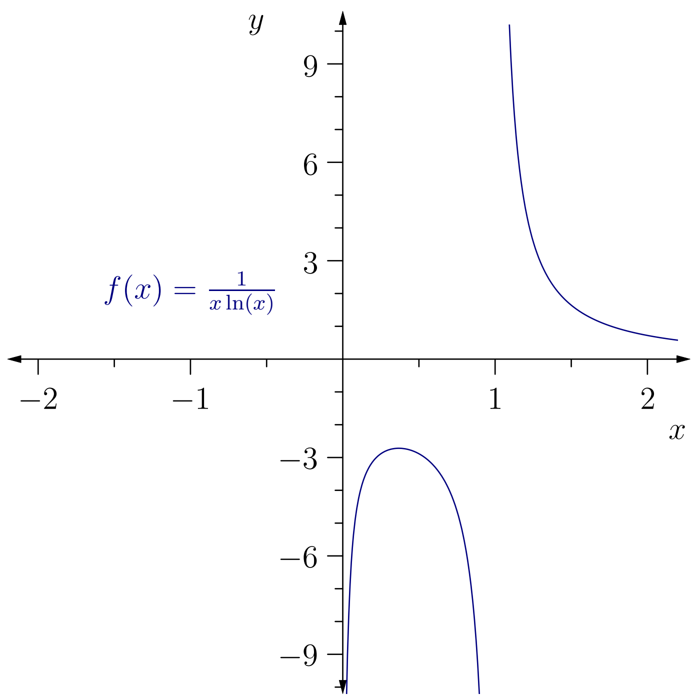

Figure 2: The function $f(x) = 1/x\ln(x)$