Domains of Functions - Example 3

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The square root of a non-negative real number x is the unique non-negative real number y such that $x = y^2$. That such a number exists is not trivial, but requires concepts we don't currently have. We label this value $y = \sqrt{x}$, and the $\sqrt{\ }$ symbol is called the *radical*. By the very nature of this definition, we do not allow the input to be negative. Indeed, if we did, we could solve the equation $x^2 + 1 = 0$ via $\sqrt{-1}$, but there is no real number such that $x^2 = -1$. Remember from the graph of the parabola that x^2 is always greater than or equal to zero. See Fig. 1 for a visual as to why x^2 is never negative, and Fig. 2 for why $x^2 + 1 = 0$ has no real solution.

Consider the following expression:

$$f(x) = \frac{\sqrt{2x}}{x - \sqrt{1 - x}} \tag{1}$$

There are several cases where $f(x)$ could fail to be well-defined. Firstly, we have a division and must ensure the denominator is non-zero. The expression $x - \sqrt{1 - x} = 0$ occurs when $x = \sqrt{1 - x}$. Squaring both sides, we must avoid $x^2 = 1 - x$, which is the same thing as avoiding $x^2 + x - 1 = 0$. Using the quadratic formula (with $a = 1$, $b = 1$, and $c = -1$), the solutions to this are:

$$x = \frac{-1 \pm \sqrt{5}}{2} \tag{2}$$

The numerator $\sqrt{2x}$ creates the restriction that $2x \geq 0$, which is equivalent to requiring $x \geq 0$. Because of this, we can rid ourselves of the requirement that $x \neq \frac{-1-\sqrt{5}}{2}$ since this value is negative and $x \geq 0$ already excludes it. Thus far we need $x \geq 0$ and $x \neq \frac{-1+\sqrt{5}}{2}$. The denominator also has the expression $\sqrt{1 - x}$. We need the input to this to be non-negative, and thus require $1 - x \geq 0$. This implies $1 \geq x$. All together we need $0 \leq x \leq 1$ and $x \neq \frac{-1+\sqrt{5}}{2}$. In set theory notation, the domain of f is:

$$D = \Big[0, \frac{-1 + \sqrt{5}}{2}\Big) \bigcup \Big(\frac{-1 + \sqrt{5}}{2}, 1\Big] \tag{3}$$

The function $f(x) = \sqrt{2x}/(x - \sqrt{1 - x})$ is plotted in Fig. 3.

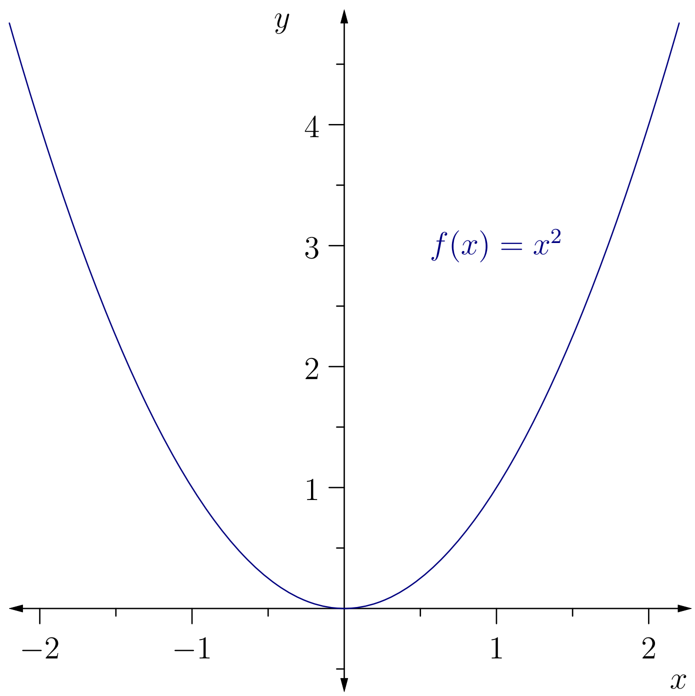

Figure 1: The graph of x^2

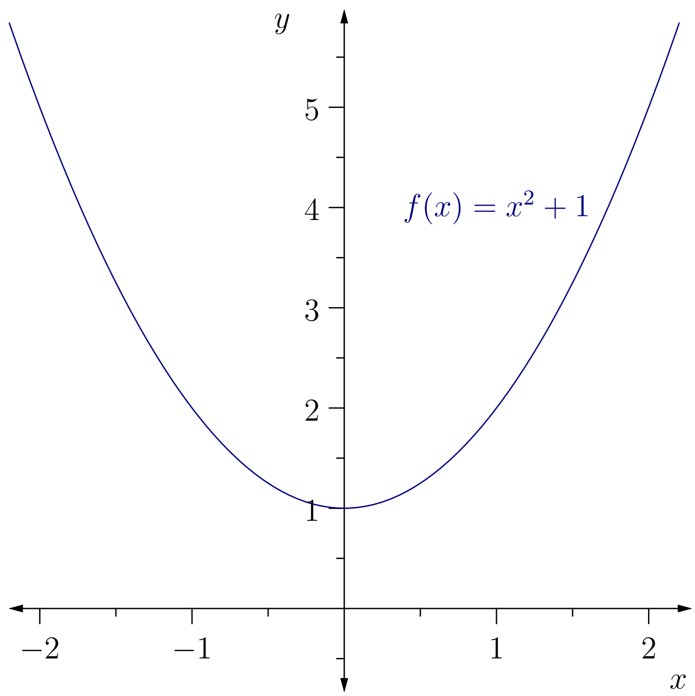

Figure 2: The graph of $x^2 + 1$

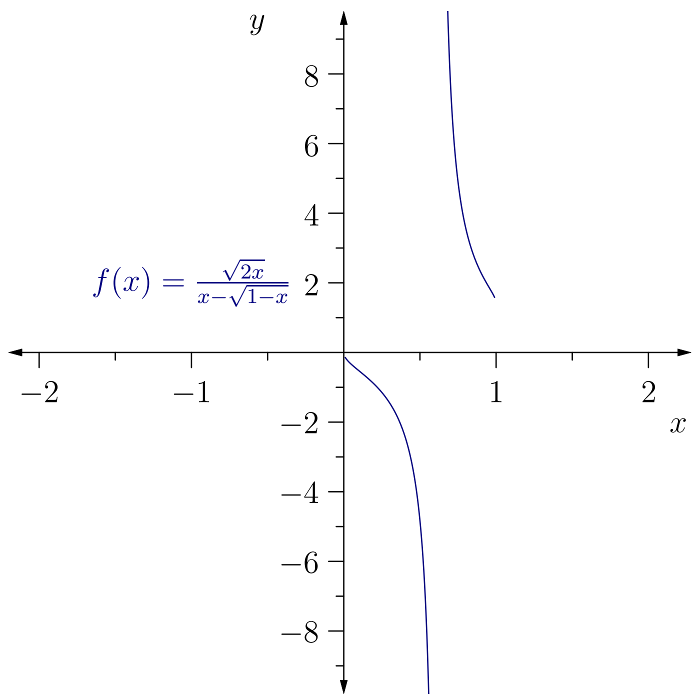

Figure 3: The graph of $f(x)$