Domains of Functions - Example 4

Ryan Maguire

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The trigonometric function sine measures the height of a point on the unit circle as a function of the angle this point makes with the x axis. From this we can see that $\sin(x)$ will lie between -1 and 1 for any input x, but we can also see that there are angles that result in 0. In particular, for every integer n (positive, negative, or zero), $\sin(n\pi) = 0$. The graph of $\sin(x)$ is shown in Fig. 1.

Consider the expression:

$$f(x) = \frac{x}{\sin(x)} \tag{1}$$

For which values is $f(x)$ well-defined? $\sin(x)$ is well-defined for any real number, so the only thing to look out for is a division-by-zero. This occurs precisely when $\sin(x) = 0$, which we've stated happens at the values $x = n\pi$ where n is any integer. So the domain is the set:

$$D = \{\, x \in \mathbb{R} \,|\, x \neq n\pi \textit{ for any } n \in \mathbb{Z} \,\} \tag{2}$$

This reads that D is the set of all real numbers x such that x is not equal to $n\pi$ for any $n \in \mathbb{Z}$. The set $\mathbb{Z}$ is the set of all integers, positive, negative, or zero. Use of the letter Z stems from the German word *Zahl*, meaning number.

If we examine Fig. 2 we see something peculiar happen at $x = 0$. Even though we have a division-by-zero, the function seems well behaved in this region. This is because the *limit* as x approaches zero for $x/\sin(x)$ is well behaved and tends towards 1. We don't yet have the machinery to prove this, but this is an example of a function such that $f(0)$ is undefined, yet the limit of $f(x)$ as x approaches 0 is well-defined.

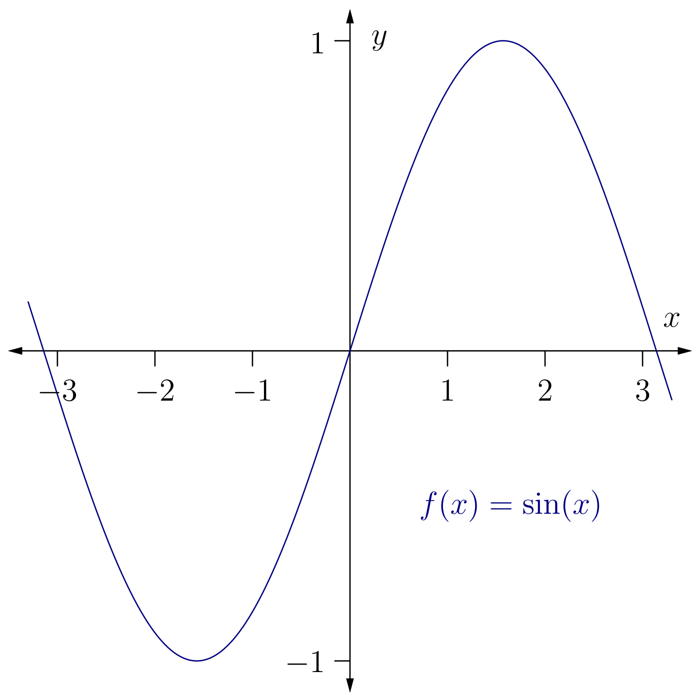

Figure 1: The sine function

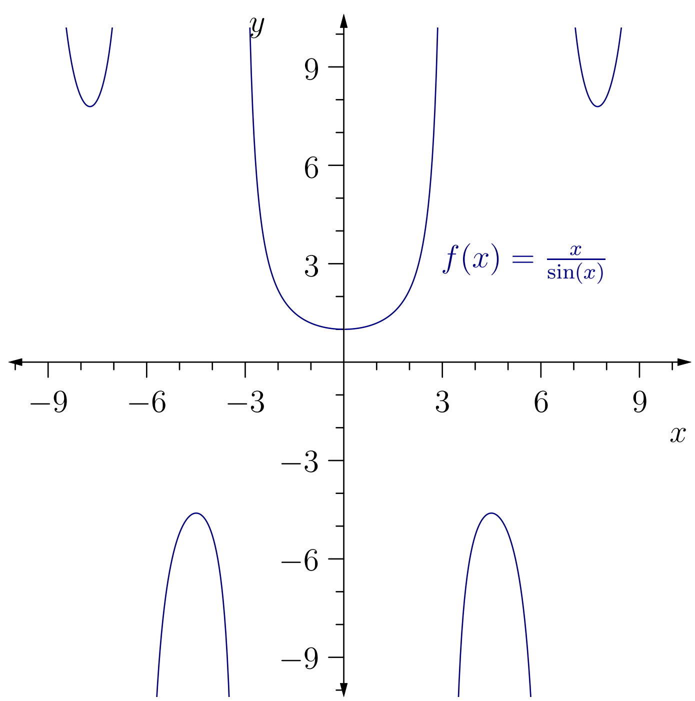

Figure 2: The function $f(x)$