$\varepsilon - \delta$ Continuity - Linear Functions

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September 28, 2023

The definition of continuity is as follows:

Definition 1 A real-valued function that is continuous at a point $x_0 \in \mathbb{R}$ is a function $f : \mathbb{R} \to \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$ with $|x - x_0| < \delta$ it is true that $|f(x) - f(x_0)| < \varepsilon$.

Let's prove linear functions $f(x) = ax + b$ are continuous ($a \neq 0$).

$$\textbf{Want:}\quad |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon \tag{1}$$

Plugging in f:

$$\textbf{Want:}\quad |x - x_0| < \delta \Rightarrow |(ax + b) - (ax_0 + b)| < \varepsilon \tag{2}$$

We can simplify this:

$$\textbf{Want:}\quad |x - x_0| < \delta \Rightarrow |ax + b - ax_0 - b| < \varepsilon \tag{3}$$

Cancel the b to get:

$$\textbf{Want:}\quad |x - x_0| < \delta \Rightarrow |ax - ax_0| < \varepsilon \tag{4}$$

Factor the a:

$$\textbf{Want:}\quad |x - x_0| < \delta \Rightarrow |a| \cdot |x - x_0| < \varepsilon \tag{5}$$

Since we only care about $|x - x_0| < \delta$, we have $|a||x - x_0| < |a|\delta$. We update our wish-list one last time:

$$\textbf{Want:}\quad |x - x_0| < \delta \Rightarrow |a|\delta \leq \varepsilon \tag{6}$$

If we made $|a|\delta \leq \varepsilon$, we would then have the following:

$$|x - x_0| < \delta \Rightarrow |a|\delta \leq \varepsilon \tag{7}$$
$$\Rightarrow |a||x - x_0| < \varepsilon \tag{8}$$
$$\Rightarrow |ax - ax_0| < \varepsilon \tag{9}$$
$$\Rightarrow |ax + b - ax_0 - b| < \varepsilon \tag{10}$$
$$\Rightarrow |(ax + b) - (ax_0 + b)| < \varepsilon \tag{11}$$

And we'd be done. All we need to do is choose a δ such that $|a| \cdot \delta \leq \varepsilon$. Since $a \neq 0$, we can simply choose $\delta = \varepsilon/|a|$. Note, any *smaller* positive value would work just fine. If you wanted to choose $\delta = \varepsilon/(2|a|)$, you could still prove $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$. There is no *best* choice of δ. The goal is just find some positive number $\delta > 0$ that works.

All of this was work to find a candidate δ. Now that we've found such a candidate, let's show that it works. Let $\varepsilon > 0$. Choose $\delta = \varepsilon/|a|$. If $|x - x_0| < \delta$, then:

$$\begin{aligned}
&|x - x_0| < \delta && \text{(Hypothesis)} \\
\Rightarrow &|x - x_0| < \frac{\varepsilon}{|a|} && \text{(Definition of } \delta\text{)} \\
\Rightarrow &|a| \cdot |x - x_0| < \varepsilon && \text{(Multiply both sides by } |a|\text{)} \\
\Rightarrow &|ax - ax_0| < \varepsilon && \text{(Distribute the } |a| \text{ using absolute value laws)} \\
\Rightarrow &|ax - ax_0 + 0| < \varepsilon && \text{(Adding zero doesn't change anything)} \\
\Rightarrow &|ax - ax_0 + (b - b)| < \varepsilon && \text{(Since } b - b = 0\text{)} \\
\Rightarrow &|ax + b - ax_0 - b| < \varepsilon && \text{(Arithmetic)} \\
\Rightarrow &|(ax + b) - (ax_0 + b)| < \varepsilon && \text{(Factor the minus sign)} \\
\Rightarrow &|f(x) - f(x_0)| < \varepsilon && \text{(Definition of } f\text{)}
\end{aligned}$$

That is, given any $\varepsilon > 0$, choose $\delta = \varepsilon/|a|$. With this, if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$, which is what we wanted to prove.