$\varepsilon - \delta$ Continuity - $f(x) = \frac{1}{x}$

Ryan Maguire

September 29, 2023

The definition of continuity is as follows:

Definition 1 A real-valued function that is continuous at a point $x_0 \in \mathbb{R}$ is a function $f : \mathbb{R} \to \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$ with $|x - x_0| < \delta$ it is true that $|f(x) - f(x_0)| < \varepsilon$.

Let's prove $f(x) = 1/x$ is continuous on $(0, \infty)$.

$$\textbf{Want:} \quad |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon \tag{1}$$

Substituting f with $f(x) = 1/x$:

$$\textbf{Want:} \quad |x - x_0| < \delta \Rightarrow \left|\frac{1}{x} - \frac{1}{x_0}\right| < \varepsilon \tag{2}$$

Simplifying the expression, this is equivalent to:

$$\textbf{Want:} \quad |x - x_0| < \delta \Rightarrow \left|\frac{x - x_0}{x x_0}\right| < \varepsilon \tag{3}$$

Which can be further simplified to:

$$\textbf{Want:} \quad |x - x_0| < \delta \Rightarrow \frac{1}{|x x_0|}|x - x_0| < \varepsilon \tag{4}$$

Since we only care about $|x - x_0| < \delta$, we have:

$$\frac{1}{|x x_0|}|x - x_0| < \frac{\delta}{|x x_0|} \tag{5}$$

So now we update our wish-list:

$$\textbf{Want:} \quad |x - x_0| < \delta \Rightarrow \frac{\delta}{|x x_0|} \leq \varepsilon \tag{6}$$

For if this were true, we would have:

$$
\begin{aligned}
|x - x_0| < \delta \Rightarrow \frac{\delta}{|xx_0|} &\leq \varepsilon && (7)\\
\Rightarrow \frac{1}{|xx_0|}|x - x_0| &< \varepsilon && (8)\\
\Rightarrow |\frac{x - x_0}{xx_0}| &< \varepsilon && (9)\\
\Rightarrow |\frac{1}{x} - \frac{1}{x_0}| &< \varepsilon && (10)
\end{aligned}
$$

So, we direct our attention to making $\delta/|xx_0| \leq \varepsilon$ a true statement. It will **not** be true for all $x \in (0, \infty)$ since as x tends to 0, the expression $1/|xx_0|$ gets arbitrarily large. But we don't need to care about values x that are really close to zero, we only care about values x that are close to x_0. So let's restrict our attention to $x \in (x_0/2, 3x_0/2)$. In other words, $|x - x_0| < x_0/2$. This is equivalent to requiring that $\delta \leq x_0/2$. If we do this, the expression $1/|xx_0|$ is now bounded. The largest this can be is when x is at it's smallest, which is $x_0/2$. Using this, we have:

$$\frac{\delta}{|xx_0|} < \frac{\delta}{|x_0^2/2|} = \frac{2\delta}{x_0^2} \tag{11}$$

We update our wish-list one last time:

$$\textbf{Want:} \quad |x - x_0| < \delta \Rightarrow \frac{2\delta}{x_0^2} \leq \varepsilon \tag{12}$$

The likely candidate is $\delta = \varepsilon x_0^2/2$, but we are already requiring that $\delta \leq x_0/2$. We need to choose the smaller of these two expressions.

We have found our candidate δ. Let's show that it works. Let $\varepsilon > 0$. Choose $\delta = \min(x_0/2, \varepsilon x_0^2/2)$. If $|x - x_0| < \delta$, then:

$$
\begin{aligned}
&|x - x_0| < \min(x_0/2, \varepsilon x_0^2/2) && \text{(Definition of } \delta\text{)}\\
\Rightarrow\ &|x - x_0| < \varepsilon x_0^2/2 && \text{(Definition of min)}\\
\Rightarrow\ &\frac{2}{x_0^2}|x - x_0| < \varepsilon && \text{(Division by a Positive Number)}
\end{aligned}
$$

But since $|x - x_0| < \min(x_0/2, \varepsilon x_0^2/2)$ we have $|x - x_0| < x_0/2$, again by the definition of min. Hence:

$$
\begin{aligned}
\frac{x_0}{2} &< x && \text{(Since } |x - x_0| < \tfrac{x_0}{2}\text{)} \\
\Rightarrow \frac{1}{x} &< \frac{2}{x_0} && \text{(Division)} \\
\Rightarrow \frac{1}{xx_0}|x - x_0| &< \frac{2}{x_0^2}|x - x_0| && \text{(Multiplication by a Positive Number)} \\
\Rightarrow \frac{1}{xx_0}|x - x_0| &< \varepsilon && \text{(Since } \tfrac{2}{x_0^2}|x - x_0| < \varepsilon\text{)} \\
\Rightarrow \left|\frac{x - x_0}{xx_0}\right| &< \varepsilon && \text{(Absolute Value Rules)} \\
\Rightarrow \left|\frac{1}{x} - \frac{1}{x_0}\right| &< \varepsilon && \text{(Simplify)} \\
\Rightarrow |f(x) - f(x_0)| &< \varepsilon && \text{(Definition of } f(x)\text{)}
\end{aligned}
$$

Therefore for all $\varepsilon > 0$ there is a $\delta > 0$ such that if $x \in (0, \infty)$ and $|x - x_0| < \delta$, then it is true that $|f(x) - f(x_0)| < \varepsilon$.