$$
\varepsilon-\delta \text { Continuity - } f(x)=\frac{1}{x}
$$

Ryan Maguire

September 29, 2023

The definition of continuity is as follows:
Definition 1 A real-valued function that is continuous at a point $x_{0} \in \mathbb{R}$ is a function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that for all $\varepsilon>0$ there exists a $\delta>0$ such that for all $x \in \mathbb{R}$ with $\left|x-x_{0}\right|<\delta$ it is true that $\left|f(x)-f\left(x_{0}\right)\right|<\varepsilon$.

Let's prove $f(x)=1 / x$ is continuous on $(0, \infty)$.

$$
\begin{equation*}
\text { Want: } \quad\left|x-x_{0}\right|<\delta \Rightarrow\left|f(x)-f\left(x_{0}\right)\right|<\varepsilon \tag{1}
\end{equation*}
$$

Substituting $f$ with $f(x)=1 / x$ :

$$
\begin{equation*}
\text { Want: } \quad\left|x-x_{0}\right|<\delta \Rightarrow\left|\frac{1}{x}-\frac{1}{x_{0}}\right|<\varepsilon \tag{2}
\end{equation*}
$$

Simplifying the expression, this is equivalent to:

$$
\begin{equation*}
\text { Want: } \quad\left|x-x_{0}\right|<\delta \Rightarrow\left|\frac{x-x_{0}}{x x_{0}}\right|<\varepsilon \tag{3}
\end{equation*}
$$

Which can be further simplified to:

$$
\begin{equation*}
\text { Want: } \quad\left|x-x_{0}\right|<\delta \Rightarrow \frac{1}{\left|x x_{0}\right|}\left|x-x_{0}\right|<\varepsilon \tag{4}
\end{equation*}
$$

Since we only care about $\left|x-x_{0}\right|<\delta$, we have:

$$
\begin{equation*}
\frac{1}{\left|x x_{0}\right|}\left|x-x_{0}\right|<\frac{\delta}{\left|x x_{0}\right|} \tag{5}
\end{equation*}
$$

So now we update our wish-list:

$$
\begin{equation*}
\text { Want: } \quad\left|x-x_{0}\right|<\delta \Rightarrow \frac{\delta}{\left|x x_{0}\right|} \leq \varepsilon \tag{6}
\end{equation*}
$$

For if this were true, we would have:

$$
\begin{align*}
\left|x-x_{0}\right|<\delta & \Rightarrow \frac{\delta}{\left|x x_{0}\right|} \leq \varepsilon  \tag{7}\\
& \Rightarrow \frac{1}{\left|x x_{0}\right|}\left|x-x_{0}\right|<\varepsilon  \tag{8}\\
& \Rightarrow\left|\frac{x-x_{0}}{x x_{0}}\right|<\varepsilon  \tag{9}\\
& \Rightarrow\left|\frac{1}{x}-\frac{1}{x_{0}}\right|<\varepsilon \tag{10}
\end{align*}
$$

So, we direct our attention to making $\delta /\left|x x_{0}\right| \leq \varepsilon$ a true statement. It will not be true for all $x \in(0, \infty)$ since as $x$ tends to 0 , the expression $1 /\left|x x_{0}\right|$ gets arbitrarily large. But we don't need to care about values $x$ that are really close to zero, we only care about values $x$ that are close to $x_{0}$. So let's restrict our attention to $x \in\left(x_{0} / 2,3 x_{0} / 2\right)$. In other words, $\left|x-x_{0}\right|<x_{0} / 2$. This is equivalent to requiring that $\delta \leq x_{0} / 2$. If we do this, the expression $1 /\left|x x_{0}\right|$ is now bounded. The largest this can be is when $x$ is at it's smallest, which is $x_{0} / 2$. Using this, we have:

$$
\begin{equation*}
\frac{\delta}{\left|x x_{0}\right|}<\frac{\delta}{\left|x_{0}^{2} / 2\right|}=\frac{2 \delta}{x_{0}^{2}} \tag{11}
\end{equation*}
$$

We update our wish-list one last time:

$$
\begin{equation*}
\text { Want: } \quad\left|x-x_{0}\right|<\delta \Rightarrow \frac{2 \delta}{x_{0}^{2}} \leq \varepsilon \tag{12}
\end{equation*}
$$

The likely candidate is $\delta=\varepsilon x_{0}^{2} / 2$, but we are already requiring that $\delta \leq x_{0} / 2$. We need to choose the smaller of these two expressions.

We have found our candidate $\delta$. Let's show that it works. Let $\varepsilon>0$. Choose $\delta=\min \left(x_{0} / 2, \varepsilon x_{0}^{2} / 2\right)$. If $\left|x-x_{0}\right|<\delta$, then:

$$
\begin{array}{rlr}
\left|x-x_{0}\right| & <\min \left(x_{0} / 2, \varepsilon x_{0}^{2} / 2\right) & \text { (Definition of } \delta \text { ) } \\
\Rightarrow\left|x-x_{0}\right| & <\varepsilon x_{0}^{2} / 2 & \text { (Definition of min) } \\
\Rightarrow \frac{2}{x_{0}^{2}}\left|x-x_{0}\right| & <\varepsilon & \text { (Division by a Positive Number) }
\end{array}
$$

But since $\left|x-x_{0}\right|<\min \left(x_{0} / 2, \varepsilon x_{0}^{2} / 2\right)$ we have $\left|x-x_{0}\right|<x_{0} / 2$, again by the
definition of min. Hence:

$$
\begin{array}{rlr}
\frac{x_{0}}{2} & <x & \\
\Rightarrow \frac{1}{x} & <\frac{2}{x_{0}} & \text { (Since }\left|x-x_{0}\right|<\frac{x_{0}}{2} \text { ) } \\
\Rightarrow \frac{1}{x x_{0}}\left|x-x_{0}\right| & <\frac{2}{x_{0}^{2}}\left|x-x_{0}\right| & \text { (Division) } \\
\Rightarrow \frac{1}{x x_{0}}\left|x-x_{0}\right|<\varepsilon & \text { (Since } \frac{2}{x_{0}^{2}}\left|x-x_{0}\right|<\varepsilon \text { ) } \\
\Rightarrow\left|\frac{x-x_{0}}{x x_{0}}\right|<\varepsilon & \text { (Absolute Value Rules) } \\
\Rightarrow\left|\frac{1}{x}-\frac{1}{x_{0}}\right|<\varepsilon & \text { (Simplify) } \\
\Rightarrow\left|f(x)-f\left(x_{0}\right)\right|<\varepsilon & \text { (Definition of } f(x) \text { ) }
\end{array}
$$

Therefore for all $\varepsilon>0$ there is a $\delta>0$ such that if $x \in(0, \infty)$ and $\left|x-x_{0}\right|<\delta$, then it is true that $\left|f(x)-f\left(x_{0}\right)\right|<\varepsilon$.

I, the copyright holder of this work, release it into the public domain. This applies worldwide. In some countries this may not be legally possible; if so: I grant anyone the right to use this work for any purpose, without any conditions, unless such conditions are required by law.

The source code used to generate this document is free software and released under version 3 of the GNU General Public License.

