$\varepsilon - \delta$ Continuity - $f(x) = \frac{1}{x}$, $x \in [0,\ \infty)$

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The definition of continuity is as follows:

Definition 1 A real-valued function that is continuous at a point $x_0 \in \mathbb{R}$ is a function $f : \mathbb{R} \to \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$ with $|x - x_0| < \delta$ it is true that $|f(x) - f(x_0)| < \varepsilon$.

Let's prove $f(x) = 1/x$ is continuous on $x \in [1, \infty)$. We want:

$$\textbf{Want:} \quad |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon \tag{1}$$

Substituting in the formula for f, we want:

$$\textbf{Want:} \quad |x - x_0| < \delta \Rightarrow \Big|\frac{1}{x} - \frac{1}{x_0}\Big| < \varepsilon \tag{2}$$

If we simplify the latter expression, we want:

$$\textbf{Want:} \quad |x - x_0| < \delta \Rightarrow \Big|\frac{x - x_0}{xx_0}\Big| < \varepsilon \tag{3}$$

And with this we've found a way to get δ in the expression. Let's simplify again:

$$\textbf{Want:} \quad |x - x_0| < \delta \Rightarrow \frac{1}{|xx_0|}|x - x_0| < \varepsilon \tag{4}$$

Since we are only looking at $|x - x_0| < \delta$, we have:

$$\frac{1}{|xx_0|}|x - x_0| < \frac{1}{|xx_0|}\delta \tag{5}$$

If we can make this last expression bounded by ε, we'd be done. That is, we update our wish-list to:

$$\textbf{Want:} \quad \frac{\delta}{|xx_0|} \leq \varepsilon \tag{6}$$

Now we recall that we are only considering $x \in [1, \infty)$, and similarly for x_0. If x and x_0 are confined to the domain $[1, \infty)$, then $1/|xx_0|$ is bounded by 1. That is, the largest $1/|xx_0|$ can be is when x and x_0 are at their smallest. And since

$x, x_0 \in [1, \infty)$, $1/|xx_0|$ is largest when $x = 1$ and $x_0 = 1$, yielding $1/|xx_0| = 1$. Because of this we can update our wish-list one last time:

$$\textbf{Want:} \quad \delta \leq \varepsilon \tag{7}$$

We can now just choose $\delta = \varepsilon$. Let's show that this works. Let $\varepsilon > 0$. Choose $\delta = \varepsilon$. If $|x - x_0| < \delta$, then:

$$\begin{aligned} |x - x_0| &< \varepsilon && \text{(Definition of } \delta\text{)} \\ \Rightarrow \frac{1}{|xx_0|}|x - x_0| &< \varepsilon && \text{(Since } \tfrac{1}{|xx_0|} \leq 1\text{)} \\ \Rightarrow \left|\frac{x - x_0}{xx_0}\right| &< \varepsilon && \text{(Absolute Value Rules)} \\ \Rightarrow \left|\frac{1}{x} - \frac{1}{x_0}\right| &< \varepsilon && \text{(Simplify)} \\ \Rightarrow |f(x) - f(x_0)| &< \varepsilon && \text{(Definition of } f\text{)} \end{aligned}$$

Hence for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in [1, \infty)$ with $|x - x_0| < \delta$ it is true that $|f(x) - f(x_0)| < \varepsilon$. That is, f is continuous at x_0.