$\varepsilon - \delta$ Continuity - Square Roots

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September 29, 2023

The definition of continuity is as follows:

Definition 1 A real-valued function that is continuous at a point $x_0 \in \mathbb{R}$ is a function $f : \mathbb{R} \to \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$ with $|x - x_0| < \delta$ it is true that $|f(x) - f(x_0)| < \varepsilon$.

Let's prove $f(x) = \sqrt{x}$ is continuous at every point $x_0 \in [0, \infty)$. First, let's handle $x_0 = 0$ separately. We want $|x - 0| < \delta$ implies $|\sqrt{x} - \sqrt{0}| < \varepsilon$. In other words, we want $|x| < \delta$ implies $|\sqrt{x}| < \varepsilon$. Choosing $\delta = \varepsilon^2$, if $x < \delta$ (we can drop the absolute value sign since $x \in [0, \infty)$, so x is never negative), then $x < \varepsilon^2$, and therefore $\sqrt{x} < \varepsilon$. Now, for $x_0 > 0$.

$$\textbf{Want:} \quad |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon \tag{1}$$

Substituting the formula for f:

$$\textbf{Want:} \quad |x - x_0| < \delta \Rightarrow |\sqrt{x} - \sqrt{x_0}| < \varepsilon \tag{2}$$

Using the trick of *conjugates* for square roots, if $\sqrt{x} + \sqrt{x_0} \neq 0$ we can write:

$$\sqrt{x} - \sqrt{x_0} = (\sqrt{x} - \sqrt{x_0})\frac{\sqrt{x} + \sqrt{x_0}}{\sqrt{x} + \sqrt{x_0}} \tag{3}$$

$$= \frac{(\sqrt{x} - \sqrt{x_0})(\sqrt{x} + \sqrt{x_0})}{\sqrt{x} + \sqrt{x_0}} \tag{4}$$

$$= \frac{x - x_0}{\sqrt{x} + \sqrt{x_0}} \tag{5}$$

Since $x_0 > 0$, we have $\sqrt{x} + \sqrt{x_0} > 0$, and so this trick is valid. Next we note for every $x \geq 0$ it is true that $\sqrt{x} \geq 0$, and therefore $\sqrt{x} + \sqrt{x_0} \geq \sqrt{x_0}$. But then:

$$\frac{1}{\sqrt{x} + \sqrt{x_0}} \leq \frac{1}{\sqrt{x_0}} \tag{6}$$

We update our wish-list:

$$\textbf{Want:} \quad |x - x_0| < \delta \Rightarrow \frac{|x - x_0|}{\sqrt{x_0}} < \varepsilon \tag{7}$$

Since we only care about $|x - x_0| < \delta$, we have:

$$\frac{|x - x_0|}{\sqrt{x_0}} < \frac{\delta}{\sqrt{x_0}} \tag{8}$$

We update our wish-list one last time:

$$\textbf{Want:} \quad |x - x_0| < \delta \Rightarrow \frac{\delta}{\sqrt{x_0}} \leq \varepsilon \tag{9}$$

Choosing $\delta = \varepsilon\sqrt{x_0}$ fulfills everything on our list.

Now that we have a candidate for δ, let's show that it works. Let $\varepsilon > 0$. Choose $\delta = \varepsilon\sqrt{x_0}$. If $|x - x_0| < \delta$, then:

$$\begin{aligned}
|x - x_0| &< \varepsilon\sqrt{x_0} && \text{(Definition of } \delta\text{)} \\
\Rightarrow \frac{|x - x_0|}{\sqrt{x_0}} &< \varepsilon && \text{(Division by a Positive Number)} \\
\Rightarrow \frac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}} &< \varepsilon && \text{(Since } \tfrac{1}{\sqrt{x}+\sqrt{x_0}} \leq \tfrac{1}{\sqrt{x_0}}\text{)} \\
\Rightarrow |\sqrt{x} - \sqrt{x_0}| &< \varepsilon && \text{(Conjugate the Expression)} \\
\Rightarrow |f(x) - f(x_0)| &< \varepsilon && \text{(Definition of } f\text{)}
\end{aligned}$$

So given $x_0 > 0$ and any $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in (0, \infty)$ with $|x - x_0| < \delta$ it is true that $|f(x) - f(x_0)| < \varepsilon$.

Before ending, let's briefly discuss *uniform* continuity. Functions such as $f(x) = ax + b$ with $x \in \mathbb{R}$ or $f(x) = 1/x$ with $x \in [1, \infty)$ have the property that, given $\varepsilon > 0$, one can choose a $\delta > 0$ that is *independent* of x_0, and such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$. For $f(x) = ax + b$, $a \neq 0$, we can choose $\delta = \varepsilon/|a|$, and for $f(x) = 1/x$ with $x \in [1, \infty)$ we can choose $\delta = \varepsilon$. Other functions, like $f(x) = x^2$ with $x \in \mathbb{R}$ or $f(x) = 1/x$ with $x \in (0, \infty)$ do not have this property. The choice of δ depends not only on ε, but on the point of interest x_0. For x^2 we get formulas like $\delta = \min(x_0/2, 5\varepsilon/2x_0)$ and for $f(x) = 1/x$ we got $\delta = \min(x_0/2, \varepsilon x_0^2/2)$. That is, the formula for δ depends on ε and x_0. This is perfectly fine within the definition of continuity. Functions where δ does *not* need to depend on x_0 are called *uniformly continuous.* Intuitively, these functions have the property that they don't get too *steep.* The function x^2 gets steeper and steeper as x increases, and so it is *not* uniformly continuous. Examining the function $f(x) = 1/x$, when we consider $x \in [1, \infty)$, the steepest the function gets is at $x = 1$. We find a δ that works at this value, and because the function is steepest there, this δ works for every other $x \in [1, \infty)$. When we look at $f(x) = 1/x$ on $(0, \infty)$, there is no *steepest point.* The function gets steeper and steeper as x gets closer to zero, and this is why δ depends on both the point of interest and ε.

With the above discussion in mind, is $f(x) = \sqrt{x}$ uniformly continuous on $x \in [0, \infty)$? We might say *no* because the formula we got is $\delta = \varepsilon\sqrt{x_0}$ and this depends on x_0. On the other hand, the *intuition* behind uniform continuity requires the function to never get too *steep*. The steepest $f(x) = \sqrt{x}$ gets is at $x_0 = 0$, but we demonstrated that the $\varepsilon - \delta$ problem works at $x_0 = 0$. So, is $\sqrt{x}$ uniformly continuous?

The answer is yes, but our method of searching for a δ did not yield anything fruitful. Let's try again. We need the following inequality: $|\sqrt{b} - \sqrt{a}| \leq \sqrt{|b-a|}$. This is proved in two steps.

Theorem 1. *If a and b are real numbers, and if $a \geq 0$ and $b \geq 0$, then:*

$$\sqrt{a+b} \leq \sqrt{a} + \sqrt{b} \tag{10}$$

Proof. Since $a \geq 0$ and $b \geq 0$, we have $0 \leq 2\sqrt{a}\sqrt{b}$. But then:

$$\begin{aligned} a + b &\leq a + 2\sqrt{a}\sqrt{b} + b && (11) \\ &= (\sqrt{a} + \sqrt{b})^2 && (12) \end{aligned}$$

And therefore $a + b \leq (\sqrt{a} + \sqrt{b})^2$. And if x and y are real numbers with $x \geq 0$, $y \geq 0$ and $x^2 \leq y^2$, then it is true that $x \leq y$. Using this, since $a + b \leq (\sqrt{a} + \sqrt{b})^2$, we have that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$. □

Theorem 2. *If a and b are real numbers, and if $a \geq 0$ and $b \geq 0$, then:*

$$|\sqrt{b} - \sqrt{a}| \leq \sqrt{|b-a|} \tag{13}$$

Proof. For simplicity, let's assume $b > a$. If $a > b$ we just need to mirror our argument, and if $a = b$ this simply says $0 \leq 0$, which is true. So assume $a < b$. We now want to prove $\sqrt{b} - \sqrt{a} \leq \sqrt{b-a}$. We'll use the previous result that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$.

$$\begin{aligned} \sqrt{b} - \sqrt{a} &= (\sqrt{b} - \sqrt{a})\frac{\sqrt{a} + \sqrt{b}}{\sqrt{a} + \sqrt{b}} && (14) \\ &= \frac{b-a}{\sqrt{a} + \sqrt{b}} && (15) \\ &\leq \frac{b-a}{\sqrt{b+a}} && (16) \\ &\leq \frac{b-a}{\sqrt{b-a}} && (17) \\ &= \sqrt{b-a} && (18) \end{aligned}$$

And therefore $\sqrt{b} - \sqrt{a} \leq \sqrt{b-a}$. □

Let's use this.

$$\textbf{Want:} \quad |x - x_0| < \delta \Rightarrow |\sqrt{x} - \sqrt{x_0}| < \varepsilon \tag{19}$$

But $|\sqrt{x} - \sqrt{x_0}| \leq \sqrt{|x - x_0|}$, so we update our wish-list:

$$\textbf{Want:} \quad |x - x_0| < \delta \Rightarrow \sqrt{|x - x_0|} < \varepsilon \tag{20}$$

Since we only care about $|x - x_0| < \delta$, we can update our wishlist once again:

$$\textbf{Want:} \quad |x - x_0| < \delta \Rightarrow \sqrt{\delta} \leq \varepsilon \tag{21}$$

We can now choose $\delta = \varepsilon^2$. Note, this is the same δ we chose for the case $x_0 = 0$. This is where $f(x) = \sqrt{x}$ is steepest, and since this δ works at $x_0 = 0$, it will work everywhere else.

Let's prove this. Let $\varepsilon > 0$. Choose $\delta = \varepsilon^2$. If $|x - x_0| < \delta$, then:

$$\begin{aligned}
|x - x_0| &< \varepsilon^2 && \text{(Definition of } \delta\text{)} \\
\Rightarrow \sqrt{|x - x_0|} &< \varepsilon && \text{(Since } \sqrt{} \text{ is an Increasing Function)} \\
\Rightarrow |\sqrt{x} - \sqrt{x_0}| &< \varepsilon && \text{(Since } |\sqrt{x} - \sqrt{x_0}| \leq \sqrt{|x - x_0|}\text{)} \\
\Rightarrow |f(x) - f(x_0)| &< \varepsilon && \text{(Definition of } f(x)\text{)}
\end{aligned}$$

And hence the square root function is *uniformly* continuous.