$\varepsilon - \delta$ Continuity - The Identity Function

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Let's show that $f(x) = x$ is continuous for all $x \in \mathbb{R}$. We'll start with the definition of continuity.

Definition 1 A real-valued function that is continuous at a point $x_0 \in \mathbb{R}$ is a function $f : \mathbb{R} \to \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$ with $|x - x_0| < \delta$ it is true that $|f(x) - f(x_0)| < \varepsilon$.

This is very wordy, but precise. Intuitively if we vary x by no more than δ from the point x_0, then $f(x)$ varies no more than ε from $f(x_0)$. That is:

Slogan
Small perturbations in x result in small perturbations in $f(x)$.

The other slogan to live by is that *nearby points go to nearby points.* Both of these phrases give intuitive meanings to continuity, but we need the formal definition to actually apply it to problems. The crucial thing to note is that continuity is defined *point-wise.* That is, a function can be continuous at one point and not the other. A function can even be continuous at only *one* point on the entire real line and discontinuous everywhere else.

Let's now show that $f(x) = x$ is continuous. This is different then problems we are used to. In elementary algebra we *solve* for variables. Now, we need to prove that no matter what $\varepsilon > 0$ is given to us, we can find a $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$. Do not get confused by the modes of thinking that apply in other areas of mathematics. We are not trying to *solve* for ε, it is given to us. Our job is to find the δ. The trick is to work backwards. Suppose we found such a $\delta > 0$. What would this say? Well, we'd have:

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon \tag{1}$$

Let's now substitute $f(x) = x$, obtaining:

$$|x - x_0| < \delta \Rightarrow |x - x_0| < \varepsilon \tag{2}$$

Now we ask ourselves *what value $\delta > 0$ would make this true?* The clear candidate is $\delta = \varepsilon$. This would then read:

$$|x - x_0| < \varepsilon \Rightarrow |x - x_0| < \varepsilon \tag{3}$$

Read aloud, *if* $|x - x_0| < \varepsilon$*, then* $|x - x_0| < \varepsilon$. This is a *tautology.* So $\delta = \varepsilon$ works. It's not the only δ we could have chosen. Indeed, any positive value less than ε would work. Suppose we chose $\delta = \varepsilon/2$. What would this say then?

$$|x - x_0| < \delta \Rightarrow |x - x_0| < \frac{\varepsilon}{2} \tag{4}$$

Well, if $|x - x_0| < \varepsilon/2$, then $|x - x_0| < \varepsilon$ since $\varepsilon/2 < \varepsilon$. That is, we would have the following chain of inequalities:

$$\begin{aligned} |x - x_0| < \delta &\Rightarrow |x - x_0| < \frac{\varepsilon}{2} < \varepsilon && (5) \\ &\Rightarrow |x - x_0| < \varepsilon && (6) \end{aligned}$$

So $\delta = \varepsilon/2$ is a valid choice. Do not get trapped in the mindset of finding the *best* δ. The choice $\delta = \varepsilon$ is, in a sense, the *best* choice since any larger value wouldn't work. But who cares? Just find a δ that works! The freedom to choose smaller values than necessary often makes the problem significantly easier.

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