$\varepsilon - \delta$ Continuity - Continuous At Only One Point

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The definition of continuity is as follows:

Definition 1 A real-valued function that is continuous at a point $x_0 \in \mathbb{R}$ is a function $f : \mathbb{R} \to \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$ with $|x - x_0| < \delta$ it is true that $|f(x) - f(x_0)| < \varepsilon$.

Let's consider the following bizarre function:

$$f(x) = \begin{cases} x & x \in \mathbb{Q} \\ -x & x \notin \mathbb{Q} \end{cases} \tag{1}$$

The set $\mathbb{Q}$ is the set of *rational numbers* of the form $x = p/q$ for integer p and q with $q \neq 0$, so if $x \notin \mathbb{Q}$ (read x *is not in* $\mathbb{Q}$), then x is irrational. So we have $f(x) = x$ for rational numbers and $f(x) = -x$ for irrational numbers. Two properties of the real number to note. First, between any two real numbers a and b with $a < b$ there is a rational number x such that $a < x < b$. Second, for any two real numbers a and b with $a < b$ there is an irrational number y such that $a < y < b$. Intuitively, the rationals and irrationals are densely distributed across the real number line. Using this, we have that if $x_0 \neq 0$, then f is *not* continuous at x_0. To see this, since the rational values give $f(x) = x$ and irrational numbers give $f(x) = -x$, in the region around x_0 there are jumps that are about $2x_0$ in height. But what would happen if we chose $x_0 = 0$?

Claim: f is continuous at 0. Let's prove this. Like every $\varepsilon - \delta$ proof, we start with the statement *let* $\varepsilon > 0$. Choose $\delta = \varepsilon$. If $|x - 0| < \delta$, then $|x| < \delta$ since $|x - 0| = |x|$. But, since $f(0) = 0$, and since $f(x) = \pm x$, depending on x, we have:

$$|f(x) - f(0)| = |f(x)| = |\pm x| = |x| < \delta = \varepsilon \tag{2}$$

And therefore $|f(x) - f(0)| < \varepsilon$. That is, f is continuous at 0.

Now ask yourself, how could we prove that f is continuous at $x = 0$ without the use of the $\varepsilon - \delta$ definition? There are other definitions of continuity (via *sequences* and via *open sets*) that are equivalent to the $\varepsilon - \delta$ but these are considerably more advanced and are reserved for courses like topology or metric spaces. So how, using what we know, would we prove f is continuous at $x = 0$? The

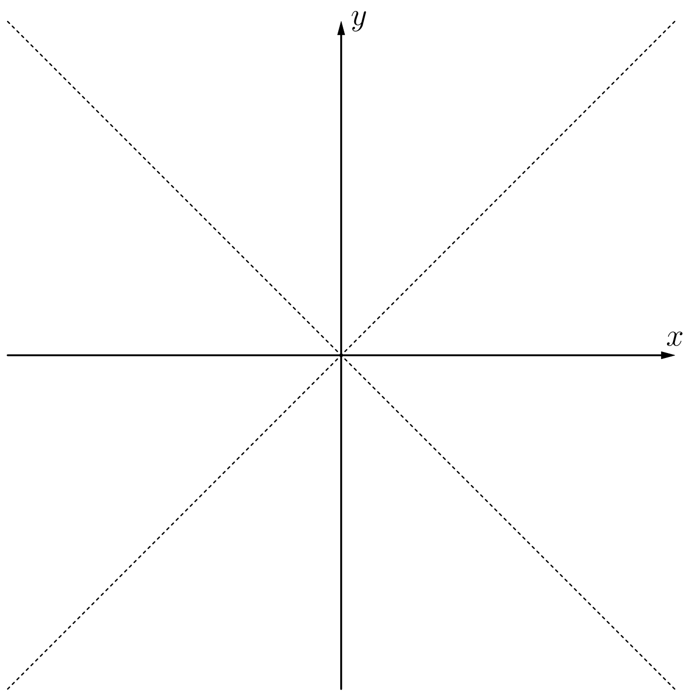

Figure 1: The graph of f

answer is, without $\varepsilon - \delta$, we most likely can't. This function is so pathological that intuition doesn't help all that much and we need to resort to a rigorous definition that we can then apply to the problem. The graph of f is shown in Fig. 1. This is a rough sketch of the function. The function itself is impossible to to draw since it has infinitely many jumps and computers can't render that. But this picture suffices for intuition.