# $\varepsilon-\delta$ Continuity - Quadratics 

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The definition of continuity is as follows:
Definition 1 A real-valued function that is continuous at a point $x_{0} \in \mathbb{R}$ is a function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that for all $\varepsilon>0$ there exists a $\delta>0$ such that for all $x \in \mathbb{R}$ with $\left|x-x_{0}\right|<\delta$ it is true that $\left|f(x)-f\left(x_{0}\right)\right|<\varepsilon$.

Let's prove $f(x)=x^{2}$ is continuous for all real numbers. Let's first work out $x_{0}=0$ so we can avoid any annoying divisions by zero that may occur in our work. For $x_{0}=0$, we have:

$$
\begin{equation*}
\text { Want: } \quad|x-0|<\delta \Rightarrow\left|x^{2}-0\right|<\varepsilon \tag{1}
\end{equation*}
$$

Simplifying, we want $|x|<\delta$ implies $|x|^{2}<\varepsilon$. If we choose $\delta=\sqrt{\varepsilon}$, we'd be done since $|x|<\delta$ implies $|x|<\sqrt{\varepsilon}$, which then implies $|x|^{2}<\varepsilon$.

Now let's prove $f(x)=x^{2}$ is continuous at any non-zero value. We have the following:

$$
\begin{equation*}
\text { Want: } \quad\left|x-x_{0}\right|<\delta \Rightarrow\left|x^{2}-x_{0}^{2}\right|<\varepsilon \tag{2}
\end{equation*}
$$

Factoring this expression on the right, we can rephrase this:

$$
\begin{equation*}
\text { Want: } \quad\left|x-x_{0}\right|<\delta \Rightarrow\left|x-x_{0}\right|\left|x+x_{0}\right|<\varepsilon \tag{3}
\end{equation*}
$$

Since we will only look at $x$ values satisfying $\left|x-x_{0}\right|<\delta$, we have:

$$
\begin{equation*}
\left|x-x_{0}\right|\left|x+x_{0}\right|<\delta\left|x+x_{0}\right| \tag{4}
\end{equation*}
$$

If we can make $\delta\left|x+x_{0}\right| \leq \varepsilon$ we'd be done, since:

$$
\begin{equation*}
\left|x^{2}-x_{0}^{2}\right|=\left|x-x_{0}\right|\left|x+x_{0}\right|<\delta\left|x+x_{0}\right| \leq \varepsilon \tag{5}
\end{equation*}
$$

So, how do we make $\delta\left|x+x_{0}\right| \leq \varepsilon$ a reality? The expression $\delta\left|x+x_{0}\right| \leq \varepsilon$ is not always true since $\left|x+x_{0}\right|$ gets arbitrarily large as $x$ gets bigger. But we don't care about larger and larger values of $x$, we only care about values of $x$ that are close to $x_{0}$. So let's suppose we only look at values that are no more than $\left|x_{0}\right| / 2$ away from $x_{0}$. That is, we are introducing the restriction that $\delta \leq\left|x_{0}\right| / 2$. If this were true, the largest $\left|x+x_{0}\right|$ could be is $5\left|x_{0}\right| / 2$. To see this, we invoke
the triangle inequality that says for any real numbers $a$ and $b$, the following is true:

$$
\begin{equation*}
|a+b| \leq|a|+|b| \tag{6}
\end{equation*}
$$

Applying this to $a=x_{0}$ and $b=x_{0} \pm x_{0} / 2$, we'd get:

$$
\begin{equation*}
\left|x+x_{0}\right| \leq|x|+\left|x_{0}\right| \leq\left|3 x_{0} / 2\right|+\left|x_{0}\right|=5\left|x_{0}\right| / 2 \tag{7}
\end{equation*}
$$

If we choose $\delta$ such that $5\left|x_{0}\right| \delta / 2 \leq \varepsilon$ we'd be done. It is tempting to write choose $\delta=2 \varepsilon / 5\left|x_{0}\right|$, but remember we've already imposed the restriction that $\delta \leq\left|x_{0}\right| / 2$. We must choose the smaller of these two.

We now have a candidate for $\delta$. Let's show that it works. Let $\varepsilon>0$. Choose $\delta=\min \left(\left|x_{0}\right| / 2,2 \varepsilon / 5\left|x_{0}\right|\right)$. If $\left|x-x_{0}\right|<\delta$, then:

$$
\begin{array}{rlr}
\left|x-x_{0}\right| & <\min \left(\frac{\left|x_{0}\right|}{2}, \frac{2 \varepsilon}{5\left|x_{0}\right|}\right) & \text { (Definition of } \delta \text { ) } \\
\Rightarrow\left|x-x_{0}\right| & <\frac{2 \varepsilon}{5\left|x_{0}\right|} & \\
\Rightarrow \frac{5\left|x_{0}\right|}{2}\left|x-x_{0}\right| & <\varepsilon & \text { (Definition of min) }
\end{array}
$$

But since $\left|x-x_{0}\right|<\min \left(\left|x_{0}\right| / 2,2 \varepsilon / 5\left|x_{0}\right|\right)$, we have $\left|x-x_{0}\right|<\left|x_{0}\right| / 2$ by the definition of min. But then:

$$
\begin{array}{rlr}
\left|x+x_{0}\right| & <\frac{5\left|x_{0}\right|}{2} & \text { (Since }\left|x-x_{0}\right|<\frac{\left|x_{0}\right|}{2} \text { ) } \\
\Rightarrow\left|x+x_{0}\right|\left|x-x_{0}\right| & <\frac{5\left|x_{0}\right|}{2}\left|x-x_{0}\right| & \text { (Multiplication by a Positive Number) } \\
\Rightarrow\left|x+x_{0}\right|\left|x-x_{0}\right| & <\varepsilon & \text { (Since } \frac{5\left|x_{0}\right|}{2}\left|x-x_{0}\right|<\varepsilon \text { ) } \\
\Rightarrow\left|x^{2}-x_{0}^{2}\right| & <\varepsilon & \text { (Simplify the Expression) } \\
\Rightarrow\left|f(x)-f\left(x_{0}\right)\right|<\varepsilon & \text { (Definition of } f \text { ) }
\end{array}
$$

And hence $f$ is continuous at $x_{0}$.

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