$\varepsilon - \delta$ Continuity - Quadratics

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The definition of continuity is as follows:

Definition 1 A real-valued function that is continuous at a point $x_0 \in \mathbb{R}$ is a function $f : \mathbb{R} \to \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$ with $|x - x_0| < \delta$ it is true that $|f(x) - f(x_0)| < \varepsilon$.

Let's prove $f(x) = x^2$ is continuous for all real numbers. Let's first work out $x_0 = 0$ so we can avoid any annoying divisions by zero that may occur in our work. For $x_0 = 0$, we have:

$$\textbf{Want:} \quad |x - 0| < \delta \Rightarrow |x^2 - 0| < \varepsilon \tag{1}$$

Simplifying, we want $|x| < \delta$ implies $|x|^2 < \varepsilon$. If we choose $\delta = \sqrt{\varepsilon}$, we'd be done since $|x| < \delta$ implies $|x| < \sqrt{\varepsilon}$, which then implies $|x|^2 < \varepsilon$.

Now let's prove $f(x) = x^2$ is continuous at any non-zero value. We have the following:

$$\textbf{Want:} \quad |x - x_0| < \delta \Rightarrow |x^2 - x_0^2| < \varepsilon \tag{2}$$

Factoring this expression on the right, we can rephrase this:

$$\textbf{Want:} \quad |x - x_0| < \delta \Rightarrow |x - x_0||x + x_0| < \varepsilon \tag{3}$$

Since we will only look at x values satisfying $|x - x_0| < \delta$, we have:

$$|x - x_0||x + x_0| < \delta|x + x_0| \tag{4}$$

If we can make $\delta|x + x_0| \leq \varepsilon$ we'd be done, since:

$$|x^2 - x_0^2| = |x - x_0||x + x_0| < \delta|x + x_0| \leq \varepsilon \tag{5}$$

So, how do we make $\delta|x + x_0| \leq \varepsilon$ a reality? The expression $\delta|x + x_0| \leq \varepsilon$ is **not** always true since $|x + x_0|$ gets arbitrarily large as x gets bigger. But we don't care about larger and larger values of x, we only care about values of x that are close to x_0. So let's suppose we only look at values that are no more than $|x_0|/2$ away from x_0. That is, we are introducing the restriction that $\delta \leq |x_0|/2$. If this were true, the largest $|x + x_0|$ could be is $5|x_0|/2$. To see this, we invoke the *triangle inequality* that says for any real numbers a and b, the following is true:

$$|a+b| \leq |a| + |b| \tag{6}$$

Applying this to $a = x_0$ and $b = x_0 \pm x_0/2$, we'd get:

$$|x + x_0| \leq |x| + |x_0| \leq |3x_0/2| + |x_0| = 5|x_0|/2 \tag{7}$$

If we choose δ such that $5|x_0|\delta/2 \leq \varepsilon$ we'd be done. It is tempting to write *choose* $\delta = 2\varepsilon/5|x_0|$, but remember we've already imposed the restriction that $\delta \leq |x_0|/2$. We must choose the smaller of these two.

We now have a candidate for δ. Let's show that it works. Let $\varepsilon > 0$. Choose $\delta = \min(|x_0|/2, 2\varepsilon/5|x_0|)$. If $|x - x_0| < \delta$, then:

$$\begin{aligned}
|x - x_0| &< \min\left(\frac{|x_0|}{2}, \frac{2\varepsilon}{5|x_0|}\right) && \text{(Definition of } \delta\text{)} \\
\Rightarrow |x - x_0| &< \frac{2\varepsilon}{5|x_0|} && \text{(Definition of min)} \\
\Rightarrow \frac{5|x_0|}{2}|x - x_0| &< \varepsilon && \text{(Multiplication by a Positive Number)}
\end{aligned}$$

But since $|x - x_0| < \min(|x_0|/2, 2\varepsilon/5|x_0|)$, we have $|x - x_0| < |x_0|/2$ by the definition of min. But then:

$$\begin{aligned}
|x + x_0| &< \frac{5|x_0|}{2} && \text{(Since } |x - x_0| < \tfrac{|x_0|}{2}\text{)} \\
\Rightarrow |x + x_0||x - x_0| &< \frac{5|x_0|}{2}|x - x_0| && \text{(Multiplication by a Positive Number)} \\
\Rightarrow |x + x_0||x - x_0| &< \varepsilon && \text{(Since } \tfrac{5|x_0|}{2}|x - x_0| < \varepsilon\text{)} \\
\Rightarrow |x^2 - x_0^2| &< \varepsilon && \text{(Simplify the Expression)} \\
\Rightarrow |f(x) - f(x_0)| &< \varepsilon && \text{(Definition of } f\text{)}
\end{aligned}$$

And hence f is continuous at x_0.