$\varepsilon - \delta$ Continuity - Continuity at a Point

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The definition of continuity is as follows:

Definition 1 A real-valued function that is continuous at a point $x_0 \in \mathbb{R}$ is a function $f : \mathbb{R} \to \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$ with $|x - x_0| < \delta$ it is true that $|f(x) - f(x_0)| < \varepsilon$.

Let's prove that $f(x) = x^2$ is continuous at $x_0 = 1$. As always, we have the following wish-list:

$$\textbf{Want:} \quad |x-1| < \delta \Rightarrow |f(x) - f(1)| < \varepsilon \tag{1}$$

Since we have an expression for f, we may as well substitute that in:

$$\textbf{Want:} \quad |x-1| < \delta \Rightarrow |x^2 - 1| < \varepsilon \tag{2}$$

Let's now search for a candidate for δ. Remember, ε is given to us. If we factor the expression $|x^2 - 1|$ we get $|(x-1)(x+1)|$, and using the product rule for the absolute value function we can simplify this to $|x-1||x+1|$. After this step we see that δ can now appear in the expression for ε. Again, we have the following wish-list:

$$\textbf{Want:} \quad |x-1| < \delta \Rightarrow |x-1||x+1| < \varepsilon \tag{3}$$

Since we're only going to look at values of x satisfying $|x-1| < \delta$, we get the following inequality:

$$|x-1||x+1| < \delta|x+1| \tag{4}$$

If we can somehow make this new expression, $\delta|x+1|$, bounded by ε, then we'd be done! That is, we'd have the following string of inequalities:

$$|x^2 - 1| = |x-1||x+1| < \delta|x+1| \leq \varepsilon \tag{5}$$

And from that we can conclude $|x^2 - 1| < \varepsilon$. So how do we make $\delta|x+1| \leq \varepsilon$ a valid inequality? The expression $|x+1|$ gets arbitrarily large as x gets bigger and bigger, and so $\delta|x+1| \leq \varepsilon$ is **not** true for all x. So what to do? Well, we don't care about all x, we only care about x values that are close to the point of interest, $x_0 = 1$ in this example. So, let's look no further than 1 away from this point to begin with. That is, we are restricting ourselves to $\delta \leq 1$. With this newly imposed restriction, we have:

$$|x-1| < \delta \Rightarrow -\delta < x - 1 < \delta \Rightarrow -1 < x - 1 < 1 \Rightarrow 0 < x < 2 \tag{6}$$

Intuitively, if x has a distance of less than 1 from the point $x_0 = 1$ on the number line, than x must be between 0 and 2. Since x is between 0 and 2, the expression $|x+1|$ is never larger than 3 (the largest it will be is when $x = 2$). In other words, since we introduced this new restriction that $\delta \leq 1$, we can now write the following inequality:

$$\delta|x+1| < 3\delta \tag{7}$$

If we can make $3\delta \leq \varepsilon$, we'd be done! It is tempting to write *choose* $\delta = \varepsilon/3$, but hold on! We already imposed $\delta \leq 1$. What if $\varepsilon/3$ is greater than this value? So we must choose the smaller of these two. Choose $\delta = \min(1, \varepsilon/3)$.

All of this was a search for a candidate δ. Now that we have such a candidate, let's show that it works. Let $\varepsilon > 0$. Choose $\delta = \min(1, \varepsilon/3)$. If $|x-1| < \delta$, then:

$$\begin{aligned}
&|x-1| < \min\left(1, \frac{\varepsilon}{3}\right) && \text{(Definition of } \delta\text{)}\\
\Rightarrow\ &|x-1| < \frac{\varepsilon}{3} && \text{(Definition of min)}\\
\Rightarrow\ &3|x-1| < \varepsilon && \text{(Multiplication by a Positive Number)}
\end{aligned}$$

But since $|x-1| < \min(1, \varepsilon/3)$, we have $|x-1| < 1$, and hence $|x+1| < 3$. But then:

$$\begin{aligned}
&|x+1||x-1| < 3|x-1| && \text{(Since } |x+1| < 3\text{)}\\
\Rightarrow\ &|x+1||x-1| < \varepsilon && \text{(Since } 3|x-1| < \varepsilon\text{)}\\
\Rightarrow\ &|x^2-1| < \varepsilon && \text{(Simplify the Expression)}\\
\Rightarrow\ &|f(x) - f(1)| < \varepsilon && \text{(Definition of } f\text{)}
\end{aligned}$$

And hence $f(x) = x^2$ is continuous at $x_0 = 1$.