Inverse Functions - Example 1

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September 29, 2023

If $f : A \to B$ is a function that takes in elements from A and returns elements in B, it is possible for there to be another function $g : B \to A$ that takes in elements from B and returns elements in A with the property that for all $a \in A$ it is true that $g(f(a)) = a$ and for all $b \in B$ it is true that $f(g(b)) = b$. Such a function is called the *inverse* of f, and is often denoted $g = f^{-1}$. There are two conditions for an inverse to exist. First, if a_0 and a_1 are elements of A, and if $f(a_0) = f(a_1)$, then $a_0 = a_1$. To see why this is required, suppose we have a function like $f(x) = x^2$, and suppose we ask what should the inverse function do at the value 1. Should $f^{-1}(1) = 1$ or should it be -1? Both have the property that they square to 1 since $1^2 = 1$ and $(-1)^2 = 1$. To prevent the inverse from being ambiguous, f must be *one-to-one*, also known as *injective*. This is precisely the statement that $f(a_0) = f(a_1)$ implies $a_0 = a_1$. Secondly, every element $b \in B$ must have some other element $a \in A$ such that $f(a) = b$. Again consider $f(x) = x^2$. What should the inverse of -1 be? There is no real number such that $x^2 = -1$, so we have an undefined value for our inverse function. Functions satisfying this second property are said to be *onto*, but also called *surjective*. So, for an inverse to exist we need the function to be *injective* and *surjective*. These two properties occur together so frequently in mathematics that they are given a new name. A *bijective* function is a function that is both *injective* and *surjective*. The function $f(x) = x^2$, when viewed as a function $f : \mathbb{R} \to \mathbb{R}$, is neither injective nor surjective.

It is often possible to restrict the domain and target of the function so that it then becomes bijective. For example, if we only consider $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, where I've used the notation $\mathbb{R}_{\geq 0}$ to mean all real numbers greater than or equal to zero, then $f(x) = x^2$ *is* bijective. Every non-negative real number squares to a unique number, and every non-negative real number is the square of a unique non-negative real number. This allows us to define the square root $f^{-1}(x) = \sqrt{x}$ which is the inverse of our original function $f(x) = x^2$. Both f and f^{-1} are shown in Fig. 1. A crucial feature to note is that $\sqrt{x}$ is the *reflection* of x^2 across the line $y = x$. This is true of inverse functions and gives us a means of plotting them, even if we can't find nice formulas.

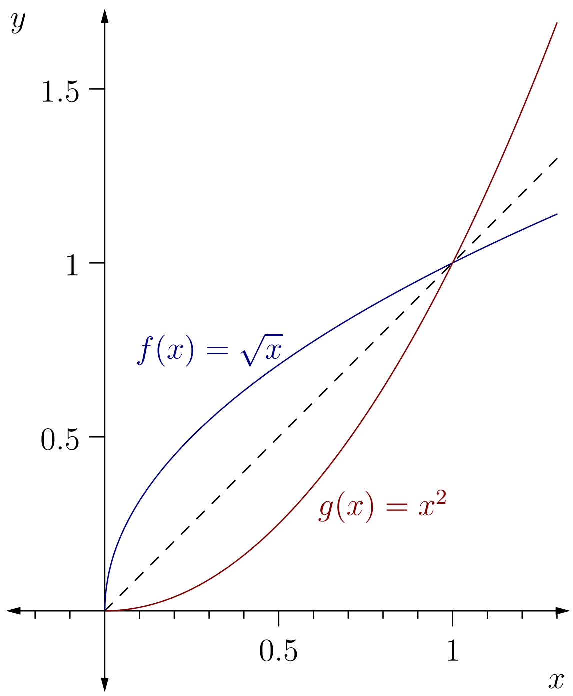

Figure 1: The graph of x^2 and $\sqrt{x}$