

## Inverse Functions - Example 2

Ryan Maguire

September 29, 2023

Consider the following expression:

$$f(x) = \frac{1}{\ln(x)} \tag{1}$$

The largest possible domain is  $(0, 1) \cup (1, \infty)$  and the range of the function is  $(-\infty, 0) \cup (0, \infty)$ . To see this, note that  $\ln(x)$  has the range of  $(-\infty, \infty)$ , so  $1/\ln(x)$  will only be missing zero in its range. That is, since for every non-zero number  $y$  there is a non-zero  $x$  such that  $y = 1/x$ , the range of  $1/\ln(x)$  will be every non-zero real number. With respect to the domain  $(0, 1) \cup (1, \infty)$  and the range  $(-\infty, 0) \cup (0, \infty)$ ,  $f$  is *bijective* meaning  $f(x_0) = f(x_1)$  implies  $x_0 = x_1$ , and for any value  $y \in (-\infty, 0) \cup (0, \infty)$  there is a value  $x$  such that  $y = f(x)$ . We won't prove this, but it can be verified from the graph of  $1/\ln(x)$  in Fig. 1. This means we can define the inverse of  $f$ . Let's try to compute what it is. To do this, we set up the equation  $y = f(x)$  and apply various operations to both sides until we've isolated  $x$  in the form  $x = g(y)$ . This function  $g$  is the inverse of  $f$ . Let's try this.

$$y = f(x) \tag{2}$$

$$y = \frac{1}{\ln(x)} \tag{3}$$

$$\frac{1}{y} = \ln(x) \tag{4}$$

$$\exp\left(\frac{1}{y}\right) = x \tag{5}$$

So, the inverse function is:

$$f^{-1}(x) = \exp\left(\frac{1}{x}\right) \tag{6}$$

The domain is  $(-\infty, 0) \cup (0, \infty)$ , which is precisely the range of  $f$ . This is not a coincidence. Like all inverse functions, the plot can be made by reflecting  $f$  across the line  $y = x$ . This is done in Fig. 2.

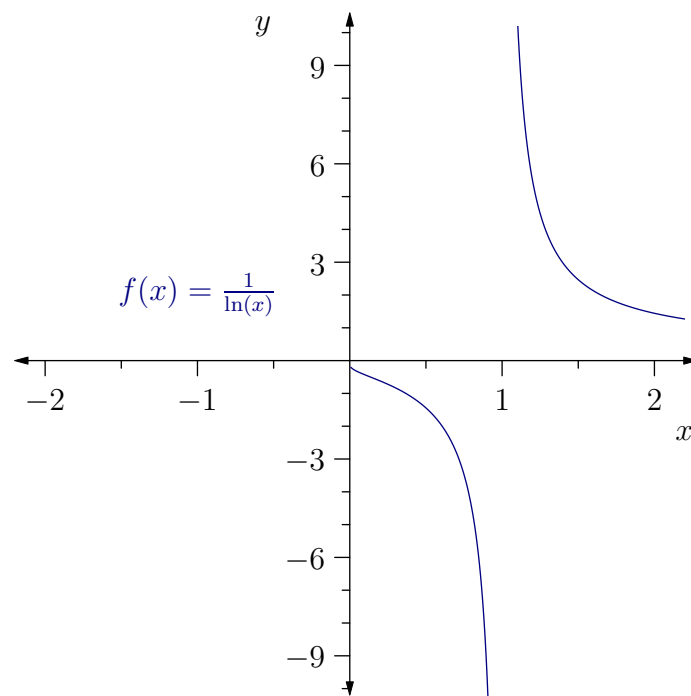


Figure 1: The function  $f(x) = 1/\ln(x)$

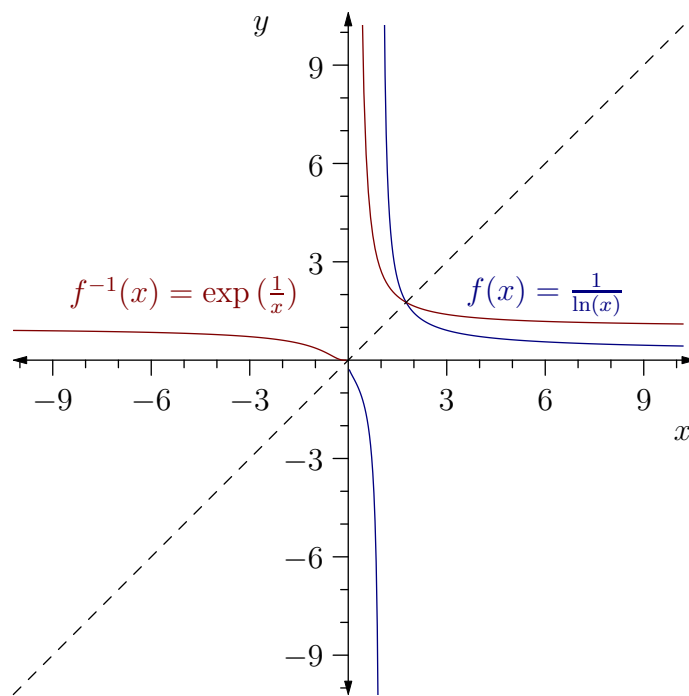


Figure 2: The function  $f(x) = 1/\ln(x)$  and it's inverse

I, the copyright holder of this work, release it into the public domain. This applies worldwide. In some countries this may not be legally possible; if so: I grant anyone the right to use this work for any purpose, without any conditions, unless such conditions are required by law.

The source code used to generate this document is free software and released under version 3 of the GNU General Public License.