Inverse Functions - Example 2

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Consider the following expression:

$$f(x) = \frac{1}{\ln(x)} \tag{1}$$

The largest possible domain is $(0,1)\cup(1,\infty)$ and the range of the function is $(-\infty,0)\cup(0,\infty)$. To see this, note that $\ln(x)$ has the range of $(-\infty,\infty)$, so $1/\ln(x)$ will only be missing zero in it's range. That is, since for every non-zero number y there is a non-zero x such that $y = 1/x$, the range of $1/\ln(x)$ will be every non-zero real number. With respect to the domain $(0,1)\cup(1,\infty)$ and the range $(-\infty,0)\cup(0,\infty)$, f is *bijective* meaning $f(x_0) = f(x_1)$ implies $x_0 = x_1$, and for any value $y \in (-\infty,0)\cup(0,\infty)$ there is a value x such that $y = f(x)$. We won't prove this, but it can be verified from the graph of $1/\ln(x)$ in Fig. 1. This means we can define the inverse of f. Let's try to compute what it is. To do this, we set up the equation $y = f(x)$ and apply various operations to both sides until we've isolated x in the form $x = g(y)$. This function g is the inverse of f. Let's try this.

$$y = f(x) \tag{2}$$

$$y = \frac{1}{\ln(x)} \tag{3}$$

$$\frac{1}{y} = \ln(x) \tag{4}$$

$$\exp\big(\frac{1}{y}\big) = x \tag{5}$$

So, the inverse function is:

$$f^{-1}(x) = \exp\big(\frac{1}{x}\big) \tag{6}$$

The domain is $(-\infty,0)\cup(0,\infty)$, which is precisely the range of f. This is not a coincidence. Like all inverse functions, the plot can be made by reflecting f across the line $y = x$. This is done in Fig. 2.

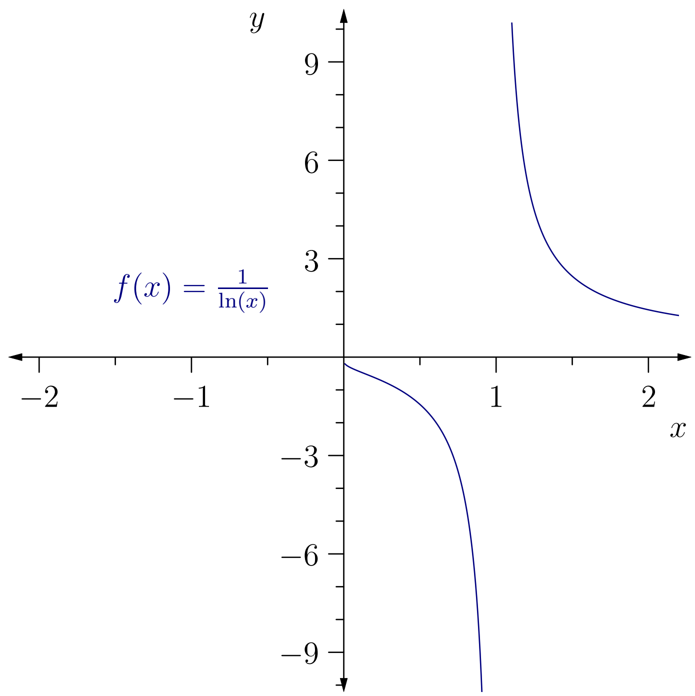

Figure 1: The function $f(x) = 1/\ln(x)$

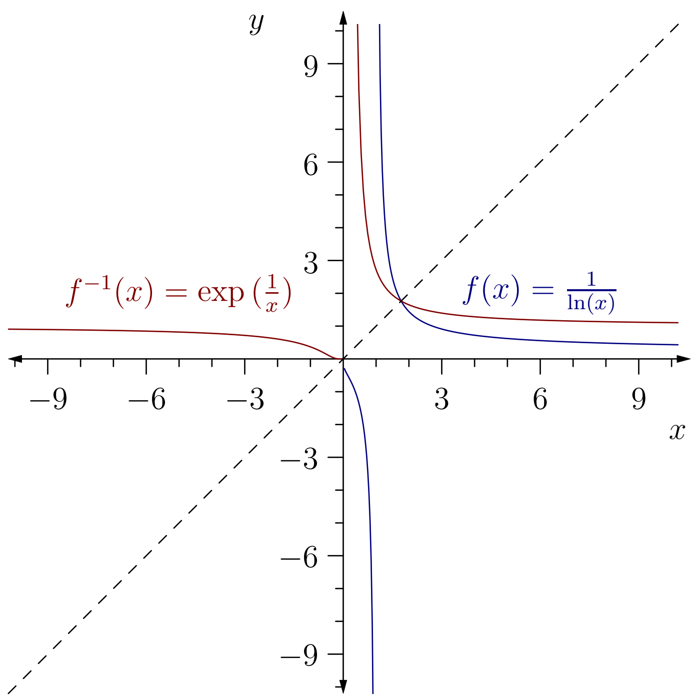

Figure 2: The function $f(x) = 1/\ln(x)$ and it's inverse