# Limits - Linear Equations 

Ryan Maguire

September 29, 2023

One of the basic laws of limits is the following identity:

$$
\begin{equation*}
\lim _{x \rightarrow x_{0}} x=x_{0} \tag{1}
\end{equation*}
$$

Let's think about what this says. As $x$ approaches $x_{0}$, we have that $x$ approaches $x_{0}$. This is a tautology. It should be intuitively clear from the graph of the function $f(x)=x$. As we pick points that get closer and closer to $x_{0}$, the value of $f$ also gets closer and closer to $x_{0}$. This is shown in Fig. 1.

The next limit law that's rather easy to believe is that the limit of a constant is the constant. That is, if $c$ is a real number, then:

$$
\begin{equation*}
\lim _{x \rightarrow x_{0}} c=c \tag{2}
\end{equation*}
$$

The function $f(x)=c$ never changes, so as $x$ varies around $x_{0}$, the value of $f(x)$ is still $c$. Graphing a constant function should convince us that this statement is true. This is done in Fig. 2.

Next, if $f$ and $g$ are two functions, if $f(x)$ tends to $a$ as $x$ approaches $x_{0}$, and if $g(x)$ tends to $b$ as $x$ approaches $x_{0}$, then the following is true:

$$
\begin{equation*}
\lim _{x \rightarrow x_{0}}(f(x)+g(x))=\lim _{x \rightarrow x_{0}} f(x)+\lim _{x \rightarrow x_{0}} g(x)=a+b \tag{3}
\end{equation*}
$$

There are a few ways to convince ourselves of this. Firstly, when we say $f(x)$ tends to $a$ we mean that $f(x)-a$ tends to zero. That is, we can make $f(x)-a$ arbitrary small as long as we choose values that are close enough to $x_{0}$. So if $f(x)-a$ and $g(x)-b$ are getting really close to zero as $x$ gets closer to $x_{0}$, let's look at what happens to $f(x)+g(x)$ :

$$
\begin{align*}
f(x)+g(x)-(a+b) & =f(x)+g(x)-a-b  \tag{4}\\
& =(f(x)-a)+(g(x)-b)  \tag{5}\\
& \approx 0+0  \tag{6}\\
& =0 \tag{7}
\end{align*}
$$

So, for values close to $x_{0}, f(x)+g(x)$ is close to $a+b$. This is precisely the statement made by Eqn. 3. This is depicted in Fig. 3.


Figure 1: The limit of $f(x)=x$ as $x$ approaches $x_{0}$

Lastly, if $c$ is a constant, and if the limit of $f$ exists as $x$ approaches $x_{0}$, then:

$$
\begin{equation*}
\lim _{x \rightarrow x_{0}} a f(x)=a \lim _{x \rightarrow x_{0}} f(x) \tag{8}
\end{equation*}
$$

That is, we can factor out constants. A visual is shown in Fig. 4. Using these four properties we can prove the limit always exists for a linear equation. Letting $f(x)=a x+b$, we have:

$$
\begin{array}{rlr}
\lim _{x \rightarrow x_{0}} f(x) & =\lim _{x \rightarrow x_{0}}(a x+b) & \\
& =\left(\lim _{x \rightarrow x_{0}} a x\right)+\left(\lim _{x \rightarrow x_{0}} b\right) & \text { (Additive Law) } \\
& =\left(\lim _{x \rightarrow x_{0}} a x\right)+b & \text { (Limit of a Constant) } \\
& =a\left(\lim _{x \rightarrow x_{0}} x\right)+b & \text { (Factoring a Constant) } \\
& =a x_{0}+b & \text { (Limit of } x \text { ) }
\end{array}
$$

So the limit of a linear function always exists and it is given by what we'd expect.


Figure 2: The limit of $f(x)=c$ as $x$ approaches $x_{0}$


Figure 3: The limit of $f(x)+g(x)$ as $x$ approaches $x_{0}$


Figure 4: The limit of $c \cdot f(x)$ as $x$ approaches $x_{0}$

I, the copyright holder of this work, release it into the public domain. This applies worldwide. In some countries this may not be legally possible; if so: I grant anyone the right to use this work for any purpose, without any conditions, unless such conditions are required by law.

The source code used to generate this document is free software and released under version 3 of the GNU General Public License.

