Limits - Linear Equations

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One of the basic laws of limits is the following identity:

$$\lim_{x \to x_0} x = x_0 \tag{1}$$

Let's think about what this says. As x approaches x_0, we have that x approaches x_0. This is a tautology. It should be intuitively clear from the graph of the function $f(x) = x$. As we pick points that get closer and closer to x_0, the value of f also gets closer and closer to x_0. This is shown in Fig. 1.

The next limit law that's rather easy to believe is that the limit of a constant is the constant. That is, if c is a real number, then:

$$\lim_{x \to x_0} c = c \tag{2}$$

The function $f(x) = c$ never changes, so as x varies around x_0, the value of $f(x)$ is still c. Graphing a constant function should convince us that this statement is true. This is done in Fig. 2.

Next, if f and g are two functions, if $f(x)$ tends to a as x approaches x_0, and if $g(x)$ tends to b as x approaches x_0, then the following is true:

$$\lim_{x \to x_0} \big(f(x) + g(x)\big) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x) = a + b \tag{3}$$

There are a few ways to convince ourselves of this. Firstly, when we say $f(x)$ tends to a we mean that $f(x) - a$ tends to zero. That is, we can make $f(x) - a$ arbitrary small as long as we choose values that are close enough to x_0. So if $f(x) - a$ and $g(x) - b$ are getting really close to zero as x gets closer to x_0, let's look at what happens to $f(x) + g(x)$:

$$f(x) + g(x) - (a + b) = f(x) + g(x) - a - b \tag{4}$$
$$= \big(f(x) - a\big) + \big(g(x) - b\big) \tag{5}$$
$$\approx 0 + 0 \tag{6}$$
$$= 0 \tag{7}$$

So, for values close to x_0, $f(x) + g(x)$ is close to $a + b$. This is precisely the statement made by Eqn. 3. This is depicted in Fig. 3.

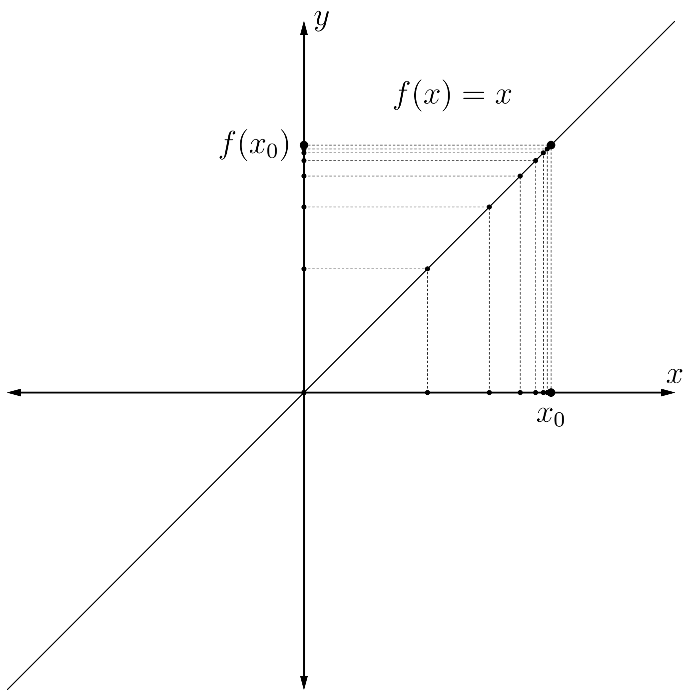

Figure 1: The limit of $f(x) = x$ as x approaches x_0

Lastly, if c is a constant, and if the limit of f exists as x approaches x_0, then:

$$\lim_{x \to x_0} af(x) = a \lim_{x \to x_0} f(x) \tag{8}$$

That is, we can factor out constants. A visual is shown in Fig. 4. Using these four properties we can prove the limit always exists for a linear equation. Letting $f(x) = ax + b$, we have:

$$\begin{aligned}
\lim_{x \to x_0} f(x) &= \lim_{x \to x_0} \big(ax + b\big) && (9)\\
&= \Big(\lim_{x \to x_0} ax\Big) + \Big(\lim_{x \to x_0} b\Big) && \text{(Additive Law)} \quad (10)\\
&= \Big(\lim_{x \to x_0} ax\Big) + b && \text{(Limit of a Constant)} \quad (11)\\
&= a\Big(\lim_{x \to x_0} x\Big) + b && \text{(Factoring a Constant)} \quad (12)\\
&= ax_0 + b && \text{(Limit of } x\text{)} \quad (13)
\end{aligned}$$

So the limit of a linear function always exists and it is given by what we'd expect.

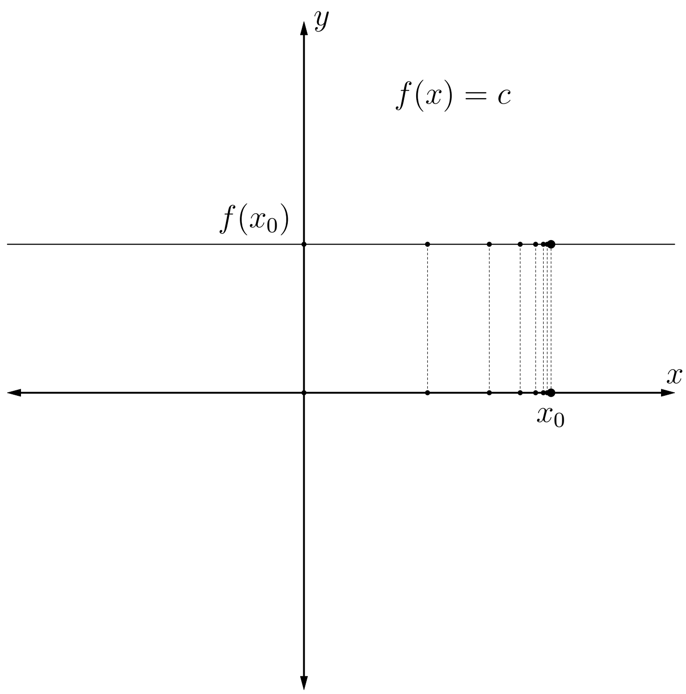

Figure 2: The limit of $f(x) = c$ as x approaches x_0

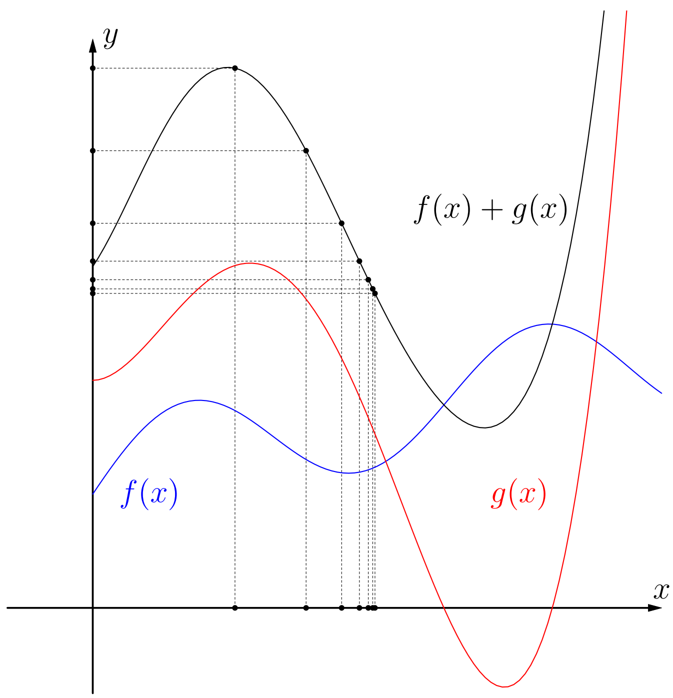

Figure 3: The limit of $f(x) + g(x)$ as x approaches x_0

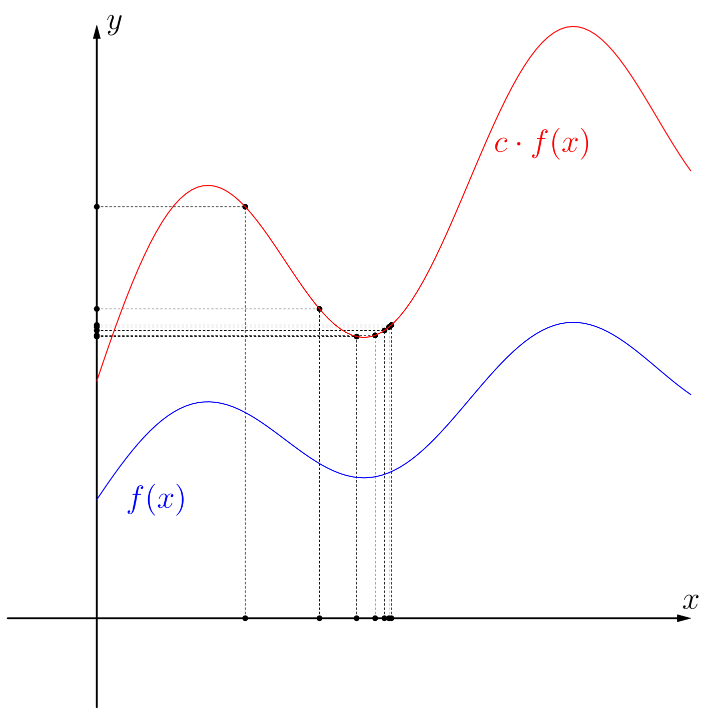

Figure 4: The limit of $c \cdot f(x)$ as x approaches x_0