# Tangent Lines - Example 1 

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September 29, 2023

Let's find the equation of the tangent line of $f(x)=2 x-3 x^{3}+x^{5}$ at $x_{0}=1$. The difference quotient for any real number $x \in \mathbb{R}$ is:

$$
\begin{equation*}
\frac{f(x+h)-f(x)}{h} \tag{1}
\end{equation*}
$$

This is the slope of the secant line passing through the points $(x, f(x))$ and $(x+h, f(x+h))$. Using $f(x)=2 x-3 x^{3}+x^{5}$ we get:

$$
\begin{equation*}
\frac{2(x+h)-3(x+h)^{3}+(x+h)^{5}-\left(2 x-3 x^{3}+x^{5}\right)}{h} \tag{2}
\end{equation*}
$$

As $h$ approaches zero, this secant line better approximates the tangent line. This is shown in Fig. 1. In fact, the limit as $h$ tends to zero is the tangent line. The limit as $h$ tends to zero is also the definition of the derivative:

$$
\begin{equation*}
\frac{\mathrm{d} f}{\mathrm{~d} x}(x)=f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} \tag{3}
\end{equation*}
$$

The notations $\frac{\mathrm{d} f}{\mathrm{~d} x}(x)$ and $f^{\prime}(x)$ are equivalent. In physics one often sees $\dot{f}(x)$ (read aloud as $f$ dot of $x$ ), and this too means the derivative of $f$ at $x$.

Let's use the sum rule for differentiation, which says that if $g_{0}$ and $g_{1}$ are differentiable functions, then:

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} x}\left(g_{0}(x)+g_{1}(x)\right)=\frac{\mathrm{d} g_{0}}{\mathrm{~d} x}(x)+\frac{\mathrm{d} g_{1}}{\mathrm{~d} x}(x) \tag{4}
\end{equation*}
$$

Applying this to $f$, we have:

$$
\begin{equation*}
\frac{\mathrm{d} f}{\mathrm{~d} x}(x)=\frac{\mathrm{d}(2 x)}{\mathrm{d} x}+\frac{\mathrm{d}\left(-3 x^{3}\right)}{\mathrm{d} x}+\frac{\mathrm{d}\left(x^{5}\right)}{\mathrm{d} x} \tag{5}
\end{equation*}
$$

Next we use the fact that constants can be factored out of the derivative. This gives us:

$$
\begin{equation*}
\frac{\mathrm{d} f}{\mathrm{~d} x}(x)=2 \frac{\mathrm{~d}(x)}{\mathrm{d} x}-3 \frac{\mathrm{~d}\left(x^{3}\right)}{\mathrm{d} x}+\frac{\mathrm{d}\left(x^{5}\right)}{\mathrm{d} x} \tag{6}
\end{equation*}
$$

To wrap this up, we apply the power rule. This says, for a function of the form $g(x)=x^{n}$, the derivative can be computed as: $g^{\prime}(x)=n x^{n-1}$. That is:

$$
\begin{equation*}
\frac{\mathrm{d}\left(x^{n}\right)}{\mathrm{d} x}=n x^{n-1} \tag{7}
\end{equation*}
$$

Using this, the derivative of $f$ becomes:

$$
\begin{align*}
\frac{\mathrm{d} f}{\mathrm{~d} x}(x) & =2 \frac{\mathrm{~d}(x)}{\mathrm{d} x}-3 \frac{\mathrm{~d}\left(x^{3}\right)}{\mathrm{d} x}+\frac{\mathrm{d}\left(x^{5}\right)}{\mathrm{d} x}  \tag{8}\\
& =2-3\left(3 x^{2}\right)+5 x^{4}  \tag{9}\\
& =2-9 x^{2}+5 x^{4} \tag{10}
\end{align*}
$$

Since we now know that $f^{\prime}(x)=2-9 x^{2}+5 x^{4}$, we can compute the slope of the tangent line of $f$ at $x_{0}=1$ by evaluating $f^{\prime}$ at 1 . We get:

$$
\begin{equation*}
f^{\prime}(1)=2-9(1)^{2}+5(1)^{4}=2-9+5=-2 \tag{11}
\end{equation*}
$$

So the slope of at $x_{0}=1$ is -2 . The tangent line has the formula:

$$
\begin{equation*}
y_{T}=m\left(x-x_{0}\right)+y_{0} \tag{12}
\end{equation*}
$$

We know the slope is $m=f^{\prime}\left(x_{0}\right)=f^{\prime}(1)=-2$, so we now have:

$$
\begin{equation*}
y_{T}=-2\left(x-x_{0}\right)+y_{0} \tag{13}
\end{equation*}
$$

When we plug in $x=x_{0}$ we see that the right hand side becomes $y_{0}$. We want the tangent line of $f$ at $x_{0}$ to have both the same slope as $f$ at $x_{0}$, and the same height. That is, we want $y_{T}$ and $f$ to meet at $x=x_{0}$. To do this, we see that we need $y_{0}=f\left(x_{0}\right)$. Since we chose $x_{0}=1$, we can compute this:

$$
\begin{equation*}
y_{0}=f\left(x_{0}\right)=f(1)=2(1)-3(1)^{3}+(1)^{5}=2-3+1=0 \tag{14}
\end{equation*}
$$

So $y_{0}=0$, and thus the tangent line is:

$$
\begin{equation*}
y_{T}=-2\left(x-x_{0}\right) \tag{15}
\end{equation*}
$$

This is plotted in Fig. 2.


Figure 1: Secant Line for $f$


Figure 2: Tangent Line for $f$

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