Tangent Lines - Example 1

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Let's find the equation of the tangent line of $f(x) = 2x - 3x^3 + x^5$ at $x_0 = 1$. The difference quotient for any real number $x \in \mathbb{R}$ is:

$$\frac{f(x+h) - f(x)}{h} \tag{1}$$

This is the slope of the *secant line* passing through the points $(x, f(x))$ and $(x+h, f(x+h))$. Using $f(x) = 2x - 3x^3 + x^5$ we get:

$$\frac{2(x+h) - 3(x+h)^3 + (x+h)^5 - \big(2x - 3x^3 + x^5\big)}{h} \tag{2}$$

As h approaches zero, this secant line better approximates the *tangent* line. This is shown in Fig. 1. In fact, the limit as h tends to zero *is* the tangent line. The limit as h tends to zero is also the definition of the derivative:

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{3}$$

The notations $\frac{\mathrm{d}f}{\mathrm{d}x}(x)$ and $f'(x)$ are equivalent. In physics one often sees $\dot{f}(x)$ (read aloud as *f dot of x*), and this too means the derivative of f at x.

Let's use the *sum rule* for differentiation, which says that if g_0 and g_1 are differentiable functions, then:

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(g_0(x) + g_1(x)\big) = \frac{\mathrm{d}g_0}{\mathrm{d}x}(x) + \frac{\mathrm{d}g_1}{\mathrm{d}x}(x) \tag{4}$$

Applying this to f, we have:

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x) = \frac{\mathrm{d}(2x)}{\mathrm{d}x} + \frac{\mathrm{d}(-3x^3)}{\mathrm{d}x} + \frac{\mathrm{d}(x^5)}{\mathrm{d}x} \tag{5}$$

Next we use the fact that constants can be factored out of the derivative. This gives us:

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x) = 2\frac{\mathrm{d}(x)}{\mathrm{d}x} - 3\frac{\mathrm{d}(x^3)}{\mathrm{d}x} + \frac{\mathrm{d}(x^5)}{\mathrm{d}x} \tag{6}$$

To wrap this up, we apply the *power rule*. This says, for a function of the form $g(x) = x^n$, the derivative can be computed as: $g'(x) = nx^{n-1}$. That is:

$$\frac{\mathrm{d}(x^n)}{\mathrm{d}x} = nx^{n-1} \tag{7}$$

Using this, the derivative of f becomes:

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}x}(x) &= 2\frac{\mathrm{d}(x)}{\mathrm{d}x} - 3\frac{\mathrm{d}(x^3)}{\mathrm{d}x} + \frac{\mathrm{d}(x^5)}{\mathrm{d}x} && (8) \\ &= 2 - 3(3x^2) + 5x^4 && (9) \\ &= 2 - 9x^2 + 5x^4 && (10) \end{aligned}$$

Since we now know that $f'(x) = 2 - 9x^2 + 5x^4$, we can compute the slope of the tangent line of f at $x_0 = 1$ by evaluating f' at 1. We get:

$$f'(1) = 2 - 9(1)^2 + 5(1)^4 = 2 - 9 + 5 = -2 \tag{11}$$

So the slope of at $x_0 = 1$ is -2. The tangent line has the formula:

$$y_T = m(x - x_0) + y_0 \tag{12}$$

We know the slope is $m = f'(x_0) = f'(1) = -2$, so we now have:

$$y_T = -2(x - x_0) + y_0 \tag{13}$$

When we plug in $x = x_0$ we see that the right hand side becomes y_0. We want the tangent line of f at x_0 to have both the same *slope* as f at x_0, and the same *height*. That is, we want y_T and f to meet at $x = x_0$. To do this, we see that we need $y_0 = f(x_0)$. Since we chose $x_0 = 1$, we can compute this:

$$y_0 = f(x_0) = f(1) = 2(1) - 3(1)^3 + (1)^5 = 2 - 3 + 1 = 0 \tag{14}$$

So $y_0 = 0$, and thus the tangent line is:

$$y_T = -2(x - x_0) \tag{15}$$

This is plotted in Fig. 2.

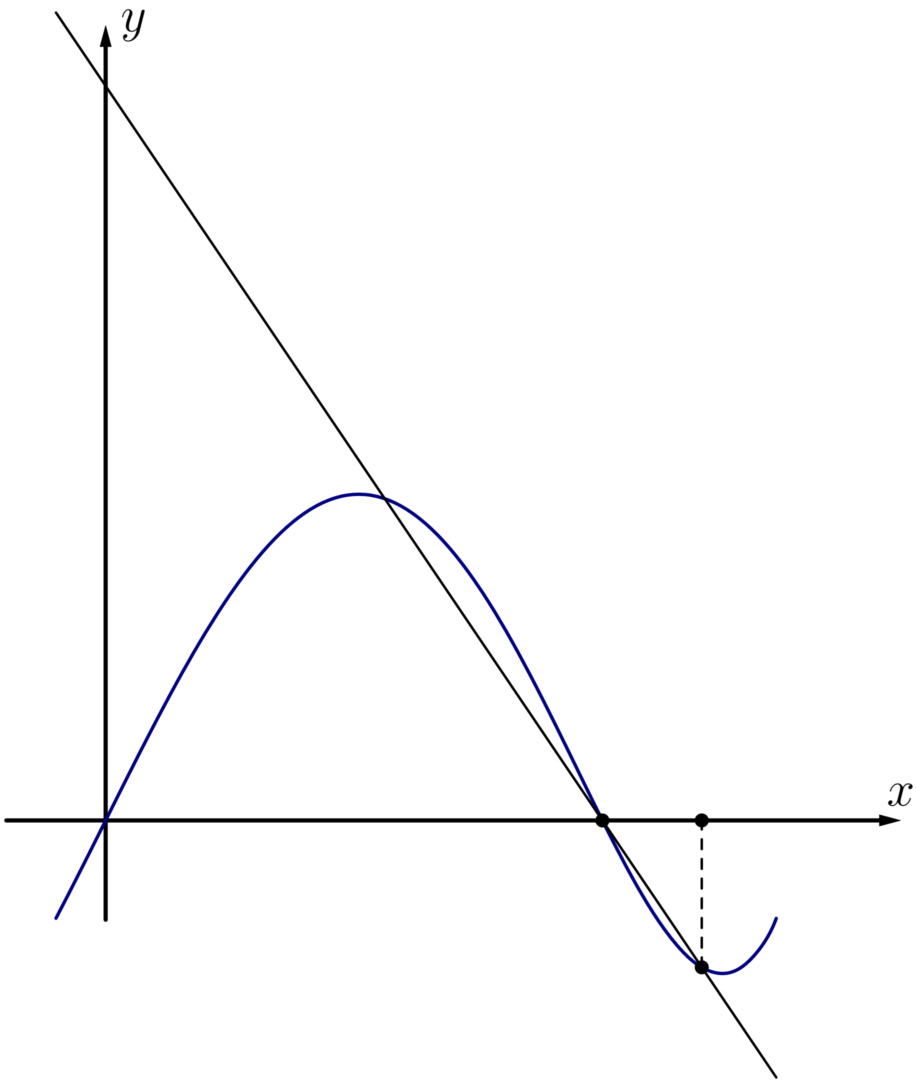

Figure 1: Secant Line for f

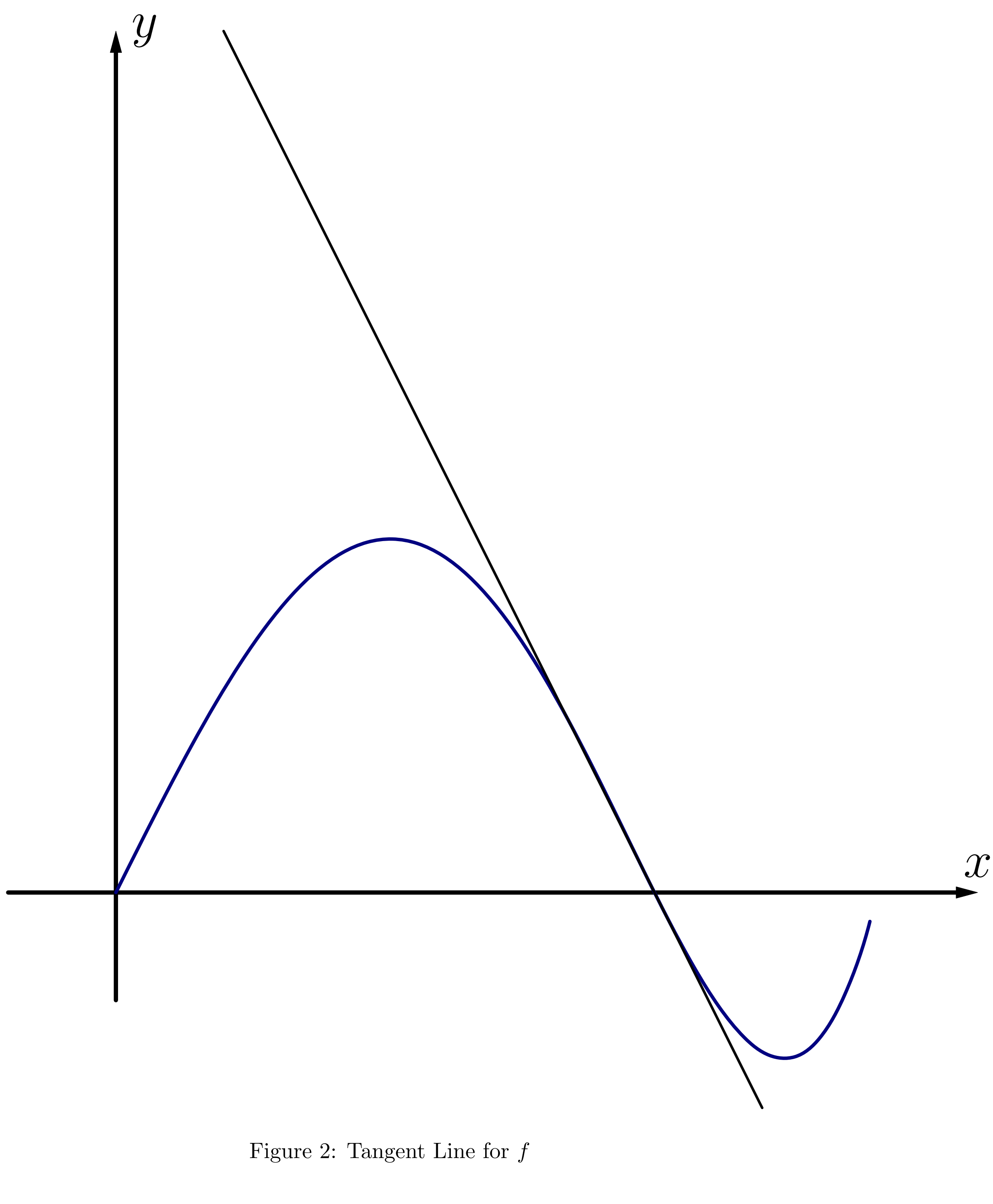

Figure 2: Tangent Line for f