# Differentiating Trigonometric Functions

Ryan Maguire

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Differentiating trigonometric functions requires two bits of information. Firstly, the trigonometric identities. Second, the small angle approximations for sine and cosine. Memorizing trigonometric formulas is, in my opinion, a bad idea. It is easy to forget if some part of the formula needs $+$ or $-$ sign, and this could ruin the rest of your work. There are two other approaches. First, you could simply consult a textbook or online reference for the required formula. This is fine. Memorizing formulas does not demonstrate mastery of mathematics, and it's silly to pretend that quick references to these equations don't exist. Most working mathematicians do not memorize these formulas.

The second way of obtaining trigonometric formulas is by *deriving* them using a quick and easy trick. This is the way many mathematicians actually incorporate these trigonometric identities into their work. Either they look them up, or they quickly derive them using the trick we're about to learn. The trick requires *complex numbers*. Complex numbers are usually taught in high school in the United States, but this may not be a global phenomenon, so here's a quick crash course in complex numbers.

A complex number is a number $z = x + iy$ where $x$ and $y$ are real numbers, and $i$ is the *imaginary unit*. Because of this, the value $x$ is called the *real part*, and the value $y$ is called the *imaginary part*. The only algebraic property we give to $i$ is that $i^2 = -1$. Just like real numbers, we can do arithmetic with complex numbers. First we ask how do we add them? We simply use some factoring.

$$(a + ib) + (c + id) = a + ib + c + id \tag{1}$$

$$= a + c + ib + id \tag{2}$$

$$= (a + c) + i(b + d) \tag{3}$$

Since $a$, $b$, $c$, and $d$ are real numbers, $a + c$ is a real number, and $b + d$ is a real number. So $(a + c) + i(b + d)$ is in the form $x + iy$ where $x$ and $y$ are real numbers, which is precisely how we defined complex numbers. Multiplication is almost as easy. We just need to use the distributive property of multiplication.

We have:

$$(a+ib)(c+id) = ac + i^2bd + iad + ibc \tag{4}$$
$$= ac + i^2bd + i(ad+bc) \tag{5}$$
$$= (ac - bd) + i(ad+bc) \tag{6}$$

Here we used the fact that $i^2 = -1$. Again we have the expression in the form $x + iy$ for real numbers $x$ and $y$. This is just about all we need to know about complex numbers in order to derive every trigonometric identity. The last bit of information needed is called *Euler's Formula*. The proof of this is not hard, but requires *Taylor Series*, which is something we don't see until the end of a calculus course. At the end of this document I'll present the proof, but if you don't care to read it that's fine too. The formula says:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{7}$$

Where $\theta$ is any real number. This formula is really bizarre! It says the exponential function and the trigonometric functions are somehow related when we use complex numbers. Now, let's use it. Suppose we want to simplify the expressions $\cos(a+b)$ and $\sin(a+b)$. How can we do this? We look at Euler's formula. Remember from exponential rules that $e^{a+b} = e^a e^b$. We can use this to simplify $\sin(a+b)$ and $\cos(a+b)$.

$$e^{i(a+b)} = \cos(a+b) + i\sin(a+b) \qquad \text{(Euler's Formula)}$$
$$e^{i(a+b)} = e^{ia+ib} \qquad \text{(Distribute } i\text{)}$$
$$= e^{ia}e^{ib} \qquad \text{(Exponential Property)}$$
$$= \big(\cos(a) + i\sin(a)\big)\big(\cos(b) + i\sin(b)\big) \qquad \text{(Euler's Formula)}$$

To simplify this last expression, we simply use the rule for multiplying complex numbers. We treat the expression like we normally would using the distributive property, but whenever we see $i^2$ we can replace it with $-1$. This gives us:

$$\big(\cos(a) + i\sin(a)\big)\big(\cos(b) + i\sin(b)\big)$$
$$= \big(\cos(a)\cos(b) - \sin(a)\sin(b)\big) + i\big(\cos(a)\sin(b) + \cos(b)\sin(a)\big) \tag{8}$$

But wait! From the very beginning, this whole thing is equal to $\cos(a+b) + i\sin(a+b)$. So we have:

$$\cos(a+b) + i\sin(a+b)$$
$$= \big(\cos(a)\cos(b) - \sin(a)\sin(b)\big) + i\big(\cos(a)\sin(b) + \cos(b)\sin(a)\big) \tag{9}$$

The real part of the left hand side is $\cos(a+b)$ and the real part of the right hand side is $\cos(a)\cos(b) - \sin(a)\sin(b)$. Since we have equality, the real parts must be equal. That is:

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b) \tag{10}$$

Similarly, the imaginary part of the left hand side is $\sin(a+b)$ and the imaginary part of the right hand side is $\cos(a)\sin(b)+\cos(b)\sin(a)$. Since we have equality, the imaginary parts must be equal. That is:

$$\sin(a+b) = \cos(a)\sin(b) + \cos(b)\sin(a) \tag{11}$$

If this last step is confusing, the fact that equality means the real and imaginary parts must be the same, consider the following. A complex number $z = x + iy$ can be thought of as a point in the plane with coordinates $(x, y)$. If we have two points in the plane $(a, b)$ and $(c, d)$, what would it mean for these points to be *equal*? They should have the same $x$ coordinate and the same $y$ coordinate, otherwise they'd be different points! That is, $(a, b) = (c, d)$ is true precisely when $a = c$ and $b = d$. Let's translate this back to complex number. $a + ib = c + id$ is true precisely when $a = c$ and $b = d$. That is, two complex numbers are equal precisely when their real parts are equal and their imaginary parts are equal.

A common problem one finds in a calculus textbook when learning about integration involves the functions $\cos^2(x)$ and $\sin^2(x)$ (particularly, the problem asks to solve $\int \cos^2(x)\mathrm{d}x$. We haven't seen integration yet, so don't worry about this). The problem becomes easier if we know the square formula for sine and cosine. Let's use Euler's method to derive it. We want to simplify $\cos^2(x)$ and $\sin^2(x)$. We have:

$$\begin{aligned}
e^{i2x} &= \cos(2x) + i\sin(2x) && \text{(Euler's Formula)}\\
e^{i2x} &= \big(e^{ix}\big)^2 && \text{(Exponential Property)}\\
\big(e^{ix}\big)^2 &= \big(\cos(x) + i\sin(x)\big)^2 && \text{(Euler's Formula)}\\
&= \big(\cos^2(x) - \sin^2(x)\big) + i2\cos(x)\sin(x) && (12)
\end{aligned}$$

Comparing the real and imaginary parts, we obtain the following formulas:

$$\cos(2x) = \cos^2(x) - \sin^2(x) \tag{13}$$

$$\sin(2x) = 2\cos(x)\sin(x) \tag{14}$$

Now you say, hold on! We wanted a formula for $\cos^2(x)$ and $\sin^2(x)$. We get that by applying the trigonometric identity $\cos^2(x) + \sin^2(x) = 1$. From this, $\sin^2(x) = 1 - \cos^2(x)$. We get:

$$\begin{aligned}
\cos(2x) &= \cos^2(x) - \sin^2(x) && (15)\\
&= \cos^2(x) - \big(1 - \cos^2(x)\big) && (16)\\
&= \cos^2(x) - 1 + \cos^2(x) && (17)\\
&= 2\cos^2(x) - 1 && (18)
\end{aligned}$$

Solving for $\cos^2(x)$, we get:

$$\cos^2(x) = \frac{\cos(2x) + 1}{2} \tag{19}$$

For $\sin^2(x)$ we again use the fact that $\sin^2(x) = 1 - \cos^2(x)$.

$$\sin^2(x) = 1 - \cos^2(x) \tag{20}$$

$$= 1 - \frac{\cos(2x) + 1}{2} \tag{21}$$

$$= \frac{1 - \cos(2x)}{2} \tag{22}$$

So, the formula for $\sin^2(x)$ is:

$$\sin^2(x) = \frac{1 - \cos(2x)}{2} \tag{23}$$

The moral of the story is that you should not feel like mathematics is a game of memorization. Memorizing tables of trigonometric functions is tedious and not very helpful, and working mathematicians do not do this! They do one of two things: Look up the desired formula, or quickly use Euler's formula to derive the result they need. With practice, using Euler's formula to calculate a certain trigonometric formula can be done very quickly in your head. If complex numbers seem too scary right now, looking up the formula in a textbook is fine.

As stated, to compute the derivative of $\cos(x)$ and $\sin(x)$ requires a few trigonometric identities, and the *small angle approximations.* These approximations are widespread in engineering and physics since they can be very accurate when used correctly, and make problems much easier to solve. If $|x|$ is small (say, less than 0.1 radians), then:

$$\sin(x) \approx x \tag{24}$$

$$\cos(x) \approx 1 - \frac{x^2}{2} \tag{25}$$

With this we can compute the derivative of $\sin(x)$. We look at the difference quotient:

$$\lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(x)\sin(h) + \cos(h)\sin(x) - \sin(x)}{h} \tag{26}$$

$$= \lim_{h\to 0} \frac{\cos(x)\sin(h)}{h} + \lim_{h\to 0} \frac{\cos(h)\sin(x) - \sin(x)}{h} \tag{27}$$

$$= \cos(x) \lim_{h\to 0} \frac{\sin(h)}{h} + \sin(x) \lim_{h\to 0} \frac{\cos(h) - 1}{h} \tag{28}$$

$$= \cos(x) \lim_{h\to 0} \frac{h}{h} + \sin(x) \lim_{h\to 0} \frac{1 - \frac{h^2}{2} - 1}{h} \tag{29}$$

$$= \cos(x) + \sin(x) \lim_{h\to 0} \frac{-h}{2} \tag{30}$$

$$= \cos(x) \tag{31}$$

The last few steps come from the small angle approximations. From this, we have:

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \cos(x) \tag{32}$$

Now let's look at $\cos(x)$.

$$
\begin{aligned}
&\lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h} && (33)\\
&= \lim_{h\to 0}\frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h} && (34)\\
&= \lim_{h\to 0}\frac{\cos(x)\cos(h)-\cos(x)}{h} - \lim_{h\to 0}\frac{\sin(x)\sin(h)}{h} && (35)\\
&= \cos(x)\lim_{h\to 0}\frac{\cos(h)-1}{h} - \sin(x)\lim_{h\to 0}\frac{\sin(h)}{h} && (36)\\
&= \cos(x)\lim_{h\to 0}\frac{1-\frac{h^2}{2}-1}{h} - \sin(x)\lim_{h\to 0}\frac{h}{h} && (37)\\
&= \cos(x)\lim_{h\to 0}\frac{-h}{2} - \sin(x) && (38)\\
&= -\sin(x) && (39)
\end{aligned}
$$

Putting this together, we have:

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = -\sin(x) \tag{40}$$

For those who want to see *why* Euler's formula is true, stick around. For those who don't, this concludes this short article. The function $e^x$ can be written as the following series:

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \tag{41}$$

$n!$ is the factorial function:

$$n! = n\cdot(n-1)\cdot(n-2)\cdots 3\cdot 2\cdot 1 \tag{42}$$

The sin and cos functions have similar formulas:

$$\cos(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \tag{43}$$

$$\sin(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} \tag{44}$$

What happens when we evaluate $e^{ix}$? We get:

$$
\begin{aligned}
e^{ix} &= \sum_{n=0}^{\infty}\frac{(ix)^n}{n!} && (45)\\
&= \sum_{n=0}^{\infty}i^n\frac{x^n}{n!} && (46)
\end{aligned}
$$

Now we need to know what happens when $i$ is raised to integer powers. By definition, $i^0 = 1$, $i^1 = i$, and $i^2 = -1$. What happens with higher powers? We have:

$$i^3 = i^2 \cdot i = -1 \cdot i = -i \tag{47}$$

We can use this new formula to compute $i^4$. We get:

$$i^4 = i^3 \cdot i = -i \cdot i = (-1) \cdot (-1) = 1 \tag{48}$$

That is, $i^4 = i^0 = 1$. The integer powers of $i$ cycle around in increments of four. Note that if $n$ is divisible by four, then $i^n = 1$, and if $n$ is even but not divisible by four, then $i^n = -1$. Similarly, the odd powers cycle between $i$ and $-i$.

Next we remark that an even number is of the form $2n$ for some integer $n$, and an odd number looks like $2n + 1$. We plug this in to our series and get:

$$e^{ix} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \tag{49}$$

But the *real* sum is the series for cos! Similarly the *imaginary* sum is the series for sin. Thus we conclude:

$$e^{ix} = \cos(x) + i\sin(x) \tag{50}$$

and Euler's formula is derived.