First Fundamental Theorem of Calculus

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The fundamental theorem of calculus relates integration and differentiation. It can be used to rigorously find the anti-derivative of various functions, and to explicitly solve the area under certain curves. The proof isn't too bad either, but first let's get some intuition as to *why* it is true. The theorem states that if $f:[a,b]\rightarrow\mathbb{R}$ is a continuous function, and if $F:[a,b]\rightarrow\mathbb{R}$ is defined by:

$$F(x)=\int_a^x f(t)\ \mathrm{d}t \tag{1}$$

Then F is differentiable, and:

$$\frac{\mathrm{d}F}{\mathrm{d}x}(x)=f(x) \tag{2}$$

Written more concisely:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\ \mathrm{d}t=f(x) \tag{3}$$

Intuitively this makes sense, we just need to understand the picture. $\mathrm{d}\int_a^x f(t)\mathrm{d}x$ asks how much more area do we get if we increment x infinitesimally so. That is, if we increment from x to $x+\mathrm{d}x$ for an infinitesimally small $\mathrm{d}x$, how much more area do we have? Well, the height at the point x times the displacement, or $f(x)\mathrm{d}x$. The value $\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\mathrm{d}t$ then asks for the ratio of this change with respect to the increment $\mathrm{d}x$. Informally we have $f(x)\mathrm{d}x/\mathrm{d}x$ and the $\mathrm{d}x$ cancel, leaving $f(x)$. So, the *rate of change* of the area under f as a function of x is simply the height of f, or $f(x)$. This geometrically makes sense (See Fig. 1). Unfortunately, all of this is very *imprecise* and may have passed as a proof in the 1600's, but not in modern times. We need to be more rigorous. We start with the *mean value theorem for integrals*.

Theorem 1. *If $f:[a,b]\rightarrow\mathbb{R}$ is a continuous function, $a<b$, then there is a point $c\in[a,b]$ such that:*

$$f(c)=\frac{1}{b-a}\int_a^b f(x)\ dx \tag{4}$$

Proof. Note that $\int_a^b f(x)\mathrm{d}x$ is just a number. Label this number A (for Area). Define $g:[a,b]\rightarrow\mathbb{R}$ to be:

$$g(x)=f(x)(b-a)-A \tag{5}$$

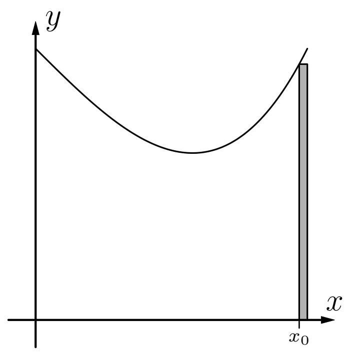

Figure 1: Intuition for the Fundamental Theorem of Calculus

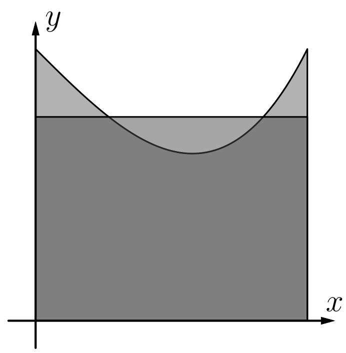

Figure 2: Drawing for the Mean Value Theorem for Integrals

Since f is continuous, g is continuous. But since f is continuous, by the extreme value theorem there are points $x_{\min}$ and $x_{\max}$ in $[a, b]$ where $f(x)$ achieves its minimum and maximum, respectively. Then $g(x_{\min})$ will be non-positive since the upper Riemann sum is always greater than or equal $f(x_{\min})(b-a)$, and $g(x_{\max})$ will be non-negative. By the intermediate value theorem, since g is continuous, there is some point $c \in [a, b]$ where $g(c) = 0$. But then:

$$f(c)(b-a) - \int_a^b f(x)\,\mathrm{d}x = 0 \tag{6}$$

Which completes the proof. □

See Fig. 2 for a visual. We can now prove the fundamental theorem of calculus.

We have:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\ \mathrm{d}t = \lim_{h\to 0}\frac{1}{h}\Big(\int_a^{x+h} f(t)\ \mathrm{d}t - \int_a^x f(t)\ \mathrm{d}t\Big) \tag{7}$$

$$= \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\ \mathrm{d}t \tag{8}$$

$$= \lim_{h\to 0}\frac{1}{h} f(c_h)(x+h-x) \tag{9}$$

$$= \lim_{h\to 0}\frac{1}{h} f(c_h)h \tag{10}$$

$$= \lim_{h\to 0} f(c_h) \tag{11}$$

Here, c_h is a value between x and $x+h$ such that:

$$f(c_h) = \frac{1}{x+h-x}\int_x^{x+h} f(t)\ \mathrm{d}t \tag{12}$$

which exists by the mean value theorem for integrals. Since $c_h \in [x, x+h]$, as h tends to zero, c_h tends to x. But f is continuous, and therefore:

$$\lim_{h\to 0} f(c_h) = f(x) \tag{13}$$

Which completes the proof.