Second Fundamental Theorem of Calculus

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September 29, 2023

The second fundamental theorem of calculus is (in my opinion) more pictorial than the first. Indeed, if you sit and think on what it is saying for a while you may convince yourself that the second fundamental theorem of calculus is *obvious*. To convince you of this, I'll need some pictures. First, the statement:

$$\int_a^b \frac{\mathrm{d}f}{\mathrm{d}x}(x)\,\mathrm{d}x = f(b) - f(a) \tag{1}$$

If you prefer f' notation, then:

$$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a) \tag{2}$$

Let's examinine what this says by approximating it with Riemann sums and difference quotients. These equations are trying to validate the following approximation:

$$\sum_{n=0}^{N-1} \frac{f(x_n + h) - f(x_n)}{h} \Delta x_n \approx f(b) - f(a) \tag{3}$$

For the derivative, we want h to be small. For the integral we want Δx_n to be small. So, why don't we just make them *equal*? Then, if one is small, the other one is too. This yields:

$$\sum_{n=0}^{N-1} \frac{f(x_n + \Delta x_n) - f(x_n)}{\Delta x_n} \Delta x_n = \sum_{n=0}^{N-1} \Big(f(x_n + \Delta x_n) - f(x_n)\Big) \tag{4}$$

Before proceeding, let's see what this means, geometrically. We start at a. We then draw the tangent line of f at a and we walk along this tangent line Δx to the right to get to our new point. Pictorially, we start at $(a, f(a))$ in the plane. We walk along the tangent line Δx and arrive at a new point $(a + \Delta x, f(a) + f'(a)\Delta x)$. We then compute the tangent line at $a + \Delta x$, walk along this line Δx to the right, and arrive at our new point. The sum over $(\Delta f/\Delta x)\Delta x$ asks *what's our change in the y axis?* As we see in the image, we end up *nearly* at the point $(b, f(b))$ meaning our net change in the y axis is *roughly* $f(b) - f(a)$. What if we make Δx smaller? With a smaller Δx we see that, after our walk, we end up very close to $(b, f(b))$. The net change in

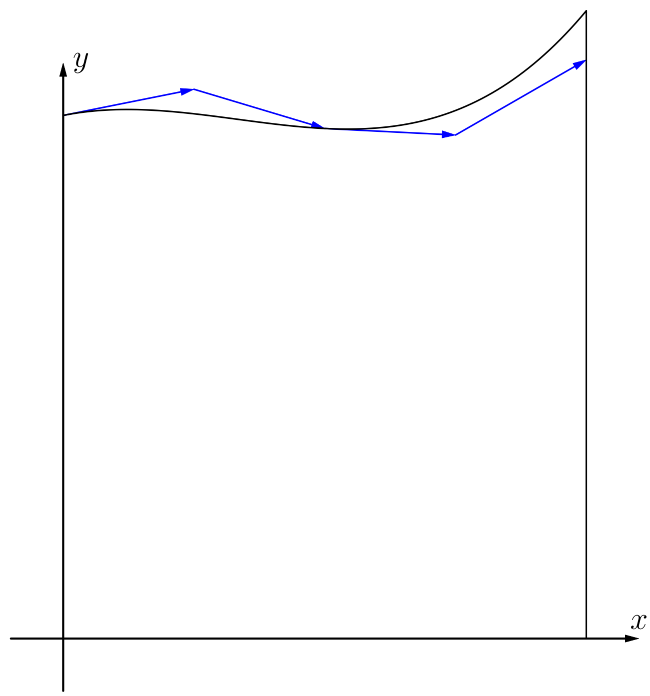

Figure 1: Approximation for Second Fundamental Theorem of Calculus

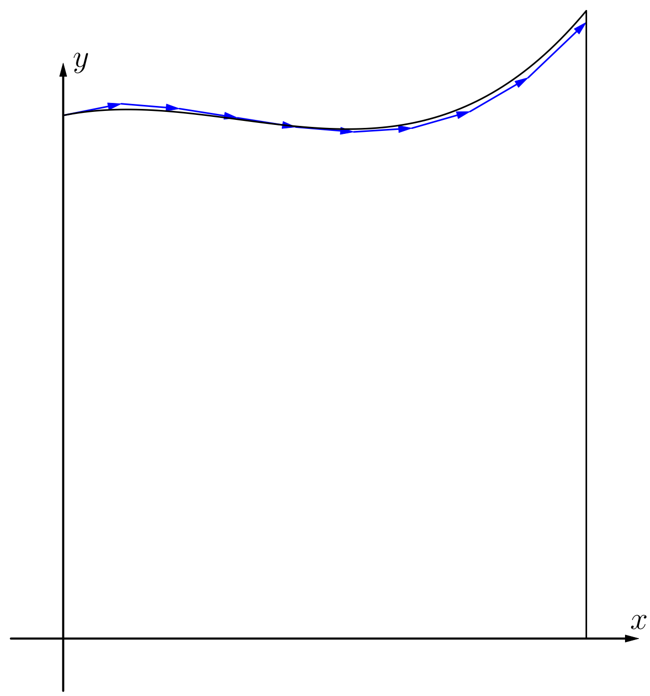

Figure 2: Approximation for Second Fundamental Theorem of Calculus

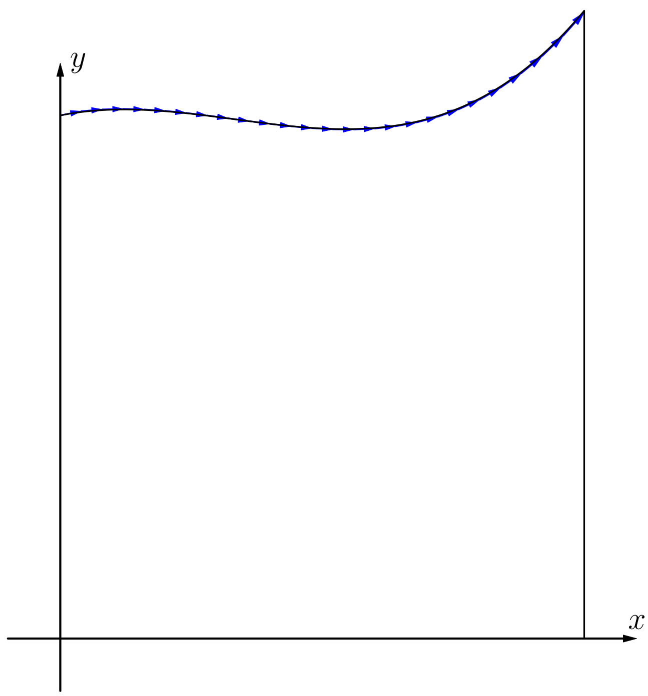

Figure 3: Approximation for Second Fundamental Theorem of Calculus

the y axis is even closer to $f(b) - f(a)$. And if we choose a really small Δx? With a really small Δx, for all intent and purpose, we end up at $(b, f(b))$ after our walk. The change in the y axis is almost identically $f(b) - f(a)$. The second fundamental theorem of calculus says that in the *limit*, we get precisely $f(b) - f(a)$. This makes sense! Remember, the integral is just a *glorified addition machine*. We are asking at each point what is the change in the y axis with respect to a change in the x axis? This is the derivative. We then sum over all of these changes. What do we get? We get the net change!

Let's phrase this in terms of physics. If we integrate our velocity $v(t)$ over a time interval $[t_0, t_1]$, what do we get? That is, we add up the instantaneous velocity $v(t)$ over all points t, what do we get? We should get our displacement! If I sum the velocity over time, I get how far I moved. That's what the second fundamental theorem of calculus says. If $r(t)$ is our position, we have:

$$\int_{t_0}^{t_1} v(t)\ \mathrm{d}t = \int_{t_0}^{t_1} r'(t)\ \mathrm{d}t = r(t_1) - r(t_0) \tag{5}$$

Let's now prove this. Given any partition x_n of the interval $[a, b]$ we have:

$$\begin{aligned}
f(b) - f(a) &= f(x_N) - f(x_0) &(6)\\
&= f(x_N) + 0 - f(x_0) &(7)\\
&= f(x_N) + \sum_{n=1}^{N-1} \Big(- f(x_n) + f(x_n)\Big) - f(x_0) &(8)\\
&= \sum_{n=0}^{N-1} \Big(f(x_{n+1}) - f(x_n)\Big) &(9)\\
&= \sum_{n=0}^{N-1} \frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n}\big(x_{n+1} - x_n\big) &(10)
\end{aligned}$$

By the mean value theorem, for each n there is a point c_n in the interval (x_n, x_{n+1}) such that:

$$f'(c_n) = \frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n} \tag{11}$$

So, we have:

$$f(b) - f(a) = \sum_{n=0}^{N-1} f'(c_n)(x_{n+1} - x_n) = \sum_{n=0}^{N-1} f'(x_n)\Delta x_n \tag{12}$$

This is true regardless of the partition we choose. So, if we have finer and finer partitions and take a limit, we get:

$$f(b) - f(a) = \lim_{N\to\infty} \sum_{n=0}^{N-1} f'(c_n)\Delta x_n = \int_a^b f'(x)\ \mathrm{d}x \tag{13}$$

Which completes the proof.