Integration by Parts

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u substitution is the reverse of the chain rule, validated by the fundamental theorem of calculus. Integration by parts is the reverse of the product rule. It says, if f and g are differentiable functions, then:

$$\int_a^b f(x)g'(x)\,\mathrm{d}x = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,\mathrm{d}x \tag{1}$$

$$= \Big(f(b)g(b) - f(a)g(a)\Big) - \int_a^b f'(x)g(x)\,\mathrm{d}x \tag{2}$$

This is often written (though, slightly abusing notation), as follows:

$$\int_a^b f\,\mathrm{d}g = fg\Big|_a^b - \int_a^b g\,\mathrm{d}f \tag{3}$$

Before we give a rigorous proof, let's give a geometric and intuitive one. Suppose f is a function with inverse f^{-1}. We can write $y = f(x)$ and $x = f^{-1}(y)$. The area under the curve f from a to b can be computed via the figure below. We compute the area under $y = f(x)$ by noting this area, plus the area to the left of $x = f^{-1}(y)$, plus the area of the grey rectangle, is equal to the area of the large rectangle with x values 0 to b and y values 0 to $f(b)$. If we know the area to the left of the curve $x = f^{-1}(y)$, we can compute the area under f as follows:

$$\int_a^b y\,\mathrm{d}x = f(b)b - f(a)a - \int_{f(a)}^{f(b)} x\,\mathrm{d}y \tag{4}$$

See the image below for a visual. This idea is best remembered via the abuse-of-notation equation. The indefinite integral is:

$$\int f\,\mathrm{d}g = fg - \int g\,\mathrm{d}f \tag{5}$$

Now you ask *what if f is not invertible?* Well, the picture isn't quite as nice. So let's prove this is true in general. The product rule says that if f and g are differentiable functions, then:

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x) \tag{6}$$

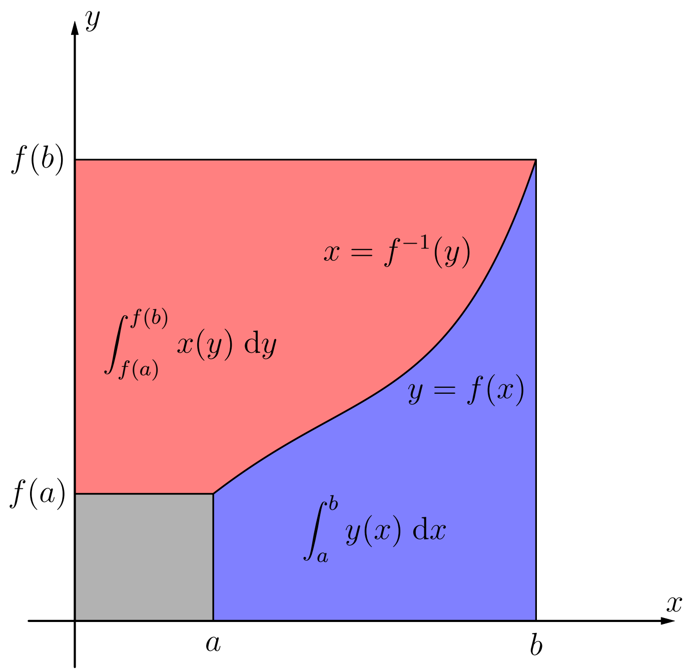

Figure 1: Visual for Integration by Parts

By the fundamental theorem of calculus, integrating gives us:

$$\int_a^b \Big(f'(x)g(x) + f(x)g'(x)\Big)\,\mathrm{d}x = \int_a^b (fg)'(x)\,\mathrm{d}x \tag{7}$$

$$= (fg)(b) - (fg)(a) \tag{8}$$

$$= f(b)g(b) - f(a)g(a) \tag{9}$$

Rearranging, we get:

$$\int_a^b f(x)g'(x)\,\mathrm{d}x = \Big(f(b)g(b) - f(a)g(a)\Big) - \int_a^b f'(x)g(x)\,\mathrm{d}x \tag{10}$$

Now let's use it. Integration by parts works well when we see products of functions. Let's integrate $x\sin(x)$. We have a choice when trying to integrate this: Do we set $f(x) = x$ and $g'(x) = \sin(x)$, or $f(x) = \sin(x)$ and $g'(x) = x$? We should set f to be the function that's gets simpler when we differentiate it. In this case, $f(x) = x$ looks ideal because differentiating gives us $f'(x) = 1$. Then we set $g'(x) = \sin(x)$ and compute the anti-derivative, which is $g(x) = -\cos(x)$.

We have:

$$\begin{aligned}
\int x\sin(x)\ \mathrm{d}x &= \int f(x)g'(x)\ \mathrm{d}x && (11)\\
&= f(x)g(x) - \int f'(x)g(x)\ \mathrm{d}x && (12)\\
&= -x\cos(x) - \int 1\big(-\cos(x)\big)\ \mathrm{d}x && (13)\\
&= -x\cos(x) + \int \cos(x)\ \mathrm{d}x && (14)\\
&= -x\cos(x) + \sin(x) + C && (15)
\end{aligned}$$

We can differentiate to verify our answer:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\Big(-x\cos(x)+\sin(x)+C\Big) &= \frac{\mathrm{d}}{\mathrm{d}x}\Big(-x\cos(x)\Big) + \frac{\mathrm{d}}{\mathrm{d}x}\Big(\sin(x)\Big) && (16)\\
&= -\cos(x) + x\sin(x) + \cos(x) && (17)\\
&= x\sin(x) && (18)
\end{aligned}$$

As expected.