Integration by Substitution

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The main techniques of evaluating integrals merely combine the rules for differentiation (in reverse) via the fundamental theorem of calculus. Let's look at the chain rule. It says if we have two differentiable functions f and g, then:

$$(g \circ f)'(x) = g'\big(f(x)\big)f'(x) \tag{1}$$

Let's integrate this. The fundamental theorem of calculus says:

$$\int_a^b (g \circ f)'(x)\ \mathrm{d}x = (g \circ f)(b) - (g \circ f)(a) \tag{2}$$

But we have a formula for $(g \circ f)'(x)$ above. Combining this, we have:

$$\int_a^b g'\big(f(x)\big)f'(x)\ \mathrm{d}x = g\big(f(b)\big) - g\big(f(a)\big) \tag{3}$$

This method of integration is usually called u substitution. Why? Well, let's set $u = f(x)$. We then have:

$$\int_a^b g'\big(f(x)\big)f'(x)\ \mathrm{d}x = \int_a^b g'(u)\frac{\mathrm{d}u}{\mathrm{d}x}\ \mathrm{d}x = g\big(f(b)\big) - g\big(f(a)\big) \tag{4}$$

But $g(f(b)) - g(f(a))$ is also equal to the following:

$$g(f(b)) - g(f(a)) = \int_{f(a)}^{f(b)} g'(u)\ \mathrm{d}u \tag{5}$$

Again, this is from the fundamental theorem of calculus. This is all rigorously justified by the chain rule and the fundamental theorem of calculus. Let's forget rigor for a second and direct our attention to the previous equation:

$$\int_a^b g'(u)\frac{\mathrm{d}u}{\mathrm{d}x}\ \mathrm{d}x = g\big(f(b)\big) - g\big(f(a)\big) \tag{6}$$

It's almost as if the $\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$ cancels and simplifies to $\mathrm{d}u$ and the limits change from a to b and become $f(a)$ to $f(b)$. This is a great way to remember this, even if it isn't a rigorous statement. We know the thinking is justified by other means.

To use the method of substitution, when trying to integrate a function f we try to break it into a product $g(u)u'$. For example, let's integrate $2x/(1+x^2)$.

$$\begin{aligned}
\int_0^1 \frac{2x}{1+x^2}\,\mathrm{d}x &= \int_0^1 \frac{1}{1+x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(1+x^2\right)\,\mathrm{d}x && (7)\\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}\ln(1+x^2)\,\mathrm{d}x && (8)\\
&= \ln(1+x^2)\big|_0^1 && (9)\\
&= \ln(1+1^2) - ln(1+0^2) && (10)\\
&= \ln(2) - \ln(1) && (11)
\end{aligned}$$

Recognizing that $2x/(1+x^2)$ is the derivative of $\ln(1+x^2)$ is hard so instead we use u substitution. I see that $2x$ is the derivative of $1+x^2$. So I set $u = 1+x^2$. Then $\mathrm{d}u = 2x\ \mathrm{d}x$. So:

$$\begin{aligned}
\int_0^1 \frac{2x}{1+x^2}\,\mathrm{d}x &= \int_0^1 \frac{2x\ \mathrm{d}x}{1+x^2} && (12)\\
&= \int_?^? \frac{\mathrm{d}u}{u} && (13)
\end{aligned}$$

What do the limits of integration become when we perform a u substitution? This comes from the chain rule. The limits go from being a to b to being $u(a)$ to $u(b)$. We have $u = 1+x^2$ so $u(0) = 1$ and $u(1) = 2$. Our integral is then:

$$\int_1^2 \frac{1}{u}\,\mathrm{d}u = \ln(u)\big|_1^2 = \ln(2) - \ln(1) \qquad (14)$$

Which is precisely what we got before.