# Point-Set Topology: Final

Summer 2022

This test is open book (any of the four recommended texts from the syllabus) and open notes (including my notes from the course site), and untimed. Collaboration with your fellow students is **not** allowed, but you can email me any clarifying questions. Use of the internet is also not allowed.

**Problem 1** (**General Topology**)

- (2 Points) Prove that a topological space $(X, \tau)$ is a Fréchet topological space if and only if for all $x \in X$ the one element set $\{ x \} \subseteq X$ is closed.
- (2 Points) Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces. Prove that $f : X \to Y$ is a homeomorphism if and only if it is a bijective continuous open mapping.
- (4 Points) Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be two topological spaces and $C(X, Y)$ the set of continuous functions between them. Prove the relation *homotopic* on $C(X, Y)$ is an equivalence relation. [Hint: You will need the pasting lemma. If $\mathcal{C}, \mathcal{D} \subseteq X$ are closed subspaces that cover $X$, if $f : \mathcal{C} \to Y$ and $g : \mathcal{D} \to Y$ are continuous, and if $f|_{\mathcal{C}\cap\mathcal{D}} = g|_{\mathcal{C}\cap\mathcal{D}}$, then the *gluing* function $h : X \to Y$ defined by:

$$h(x) = \begin{cases} f(x) & x \in \mathcal{C} \\ g(x) & x \in \mathcal{D} \end{cases} \tag{1}$$

  is continuous. You do not need to prove the pasting lemma.]

**Problem 2** (**Separation Properties**)

- (2 Points) A completely Hausdorff space is a topological space $(X,\,\tau)$ such that for all distinct points $x, y \in X$ there is a continuous function $f : X \to [0,\,1]$ (where $[0,\,1]$ inherits the subspace topology from $\mathbb{R}$) such that $f(x) = 0$ and $f(y) = 1$. Prove that a completely Hausdorff space is Hausdorff.

- (2 Points) This does not reverse, in general. The prime integer topology on $\mathbb{N}$ is Hausdorff but not completely Hausdorff. Prove that if $(X,\,\tau)$ is Hausdorff and normal, then it is completely Hausdorff.

- (4 Points) A $G_\delta$ set in a topological space $(X,\,\tau)$ is a set $A \subseteq X$ such that there is a sequence $\mathcal{U} : \mathbb{N} \to \tau$ of open sets such that $A = \bigcap_{n\in\mathbb{N}} \mathcal{U}_n$. A $G_\delta$ space is a topological space $(X,\,\tau)$ such that for all closed subsets $\mathcal{C} \subseteq X$ it is true that $\mathcal{C}$ is a $G_\delta$ set. Prove that if $(X,\,\tau)$ is a normal $G_\delta$ space, then it is perfectly normal. [Hint: You will need the uniform convergence theorem. Given a sequence of continuous functions $f_n : X \to [0,\,1]$, the sum $g : X \to [0,\,1]$ defined by:

$$g(x) = \sum_{n=0}^{\infty} \frac{f_n(x)}{2^{n+1}} \tag{2}$$

is continuous. You can use this freely. We also proved that a space is perfectly normal if and only if for all closed $\mathcal{C} \subseteq X$ there is a continuous function $h : X \to [0,\,1]$ such that $\mathcal{C} = h^{-1}[\{\,0\,\}]$. That is, you do not need to consider another closed set $\mathcal{D}$ and try to simultaneously force $\mathcal{D} = h^{-1}[\{\,1\,\}]$ to be true. You may also use this freely.]

**Problem 3 (Compactness)**

- (2 Points) Let $(X, \tau_X)$ be compact and $(Y, \tau_Y)$ Hausdorff. Prove that a continuous bijection $f : X \to Y$ is a homeomorphism.
- (2 Points) Prove that if $(X, \tau_X)$ and $(Y, \tau_Y)$ are compact, then $(X \times Y, \tau_{X\times Y})$ is compact.
- (2 Points) Let $(X, \tau_X)$ be a compact topological space, $(Y, \tau_Y)$ a topological space, and $q : X \to Y$ a quotient map. Prove that $(Y, \tau_Y)$ is compact.

**Problem 4 (Connectedness)**

- (4 Points) Prove that a path connected space $(X, \tau)$ is connected.
- (4 Points) Prove the intermediate value theorem. If $a, b \in \mathbb{R}$, $a < b$, and if $f : [a, b] \to \mathbb{R}$ is continuous, $f(a) < f(b)$, then for all $y \in \big(f(a), f(b)\big)$ there is an $x \in (a, b)$ such that $f(x) = y$.

**Problem 5** (**Paracompactness and Manifolds**)

- (4 Points) Prove Dieudonné's theorem. A paracompact Hausdorff space is normal. [Hint: We proved paracompact Hausdorff spaces are regular, and then used some *hand-waving* due to lack of time to prove paracompact Hausdorff spaces are normal. Your job is to generalize the argument for regularity and make a formal proof.]
- (4 Points) The torus $\mathbb{T}^2$ is the product of the circle with itself. Using stereographic projection we covered $\mathbb{S}^1$ with two charts, meaning $\mathbb{S}^1 \times \mathbb{S}^1$ can be covered with four charts. Show that you can cover $\mathbb{T}^2$ with just three charts (you can actually do it with just two). Show that you can **not** cover it with just one chart.