Point-Set Topology: Midterm

Summer 2022

Problem 1 (**Logic**)

The truth table for a logical connective (such as $\Rightarrow$) that combines two propositions P and Q into a single proposition (like $P \Rightarrow Q$) is a table that exhausts all possibilities of P and Q being true and false. The truth table for implication is given in Tab. 1

P	Q	$P \Rightarrow Q$
False	False	True
False	True	True
True	False	False
True	True	True

Table 1: Truth Table for Implication

Prove the absorption laws. If P and Q are propositions, then P *or* (P *and* Q) if and only if P. Using $\vee$ and $\wedge$ this says:

$$P \vee (P \wedge Q) \Leftrightarrow P \tag{1}$$

Also, P *and* (P *or* Q) if and only if P. That is:

$$P \wedge (P \vee Q) \Leftrightarrow P \tag{2}$$

- (1 Point) Construct the truth table for $P \vee Q$.
- (1 Point) Construct the truth table for $P \wedge Q$.
- (1 Point) Construct the truth table for $P \vee (P \wedge Q)$.
- (1 Point) Construct the truth table for $P \wedge (P \vee Q)$.
- (1 Point) Compare these with P to prove the absorption laws.

Prove that implication can be defined by *negation* ($\neg$) and *logical or* ($\vee$).

- (1 Point) Give the truth table for $\neg P$.
- (1 Point) Give the truth table for $(\neg P) \vee Q$.
- (1 Point) Compare this with implication $P \Rightarrow Q$.

Problem 2 (Set Theory)

Here you will construct the real numbers.

- (2 Points) Show that if A and B are sets, there is a set of all functions $f : A \to B$. [Hint: Functions are subsets $f \subseteq A \times B$ with a special property. Use the axiom of the power set and the axiom schema of specification to construct the set of all functions from A to B.]

- (1 Point) Given the rational numbers $\mathbb{Q}$ with the standard metric $d(x, y) = |x - y|$, state the definition of a Cauchy sequence in $\mathbb{Q}$.

- (2 Points) Let A be the set of all Cauchy sequences $a : \mathbb{N} \to \mathbb{Q}$ (This set exists by part 1 of this problem). Define the relation R on A by aRb if and only if $|a_n - b_n| \to 0$. Prove R is an equivalence relation.

- (3 Points) Let $\mathbb{R} = A/R$. Define $+$ on $\mathbb{R}$ by $[a] + [b] = [c]$ where $c : \mathbb{N} \to \mathbb{Q}$ is the sequence $c_n = a_n + b_n$. Show that $c : \mathbb{N} \to \mathbb{Q}$ is indeed a Cauchy sequence and that $+$ is well defined on $\mathbb{R}$.

Problem 3 (**Metric Spaces**)

- (1 Point) State the definition of a metric space.
- (1 Point) State the definition of a convergent sequence.
- (1 Point) State the definition of a continuous function from a metric space (X, d_X) to a metric space (Y, d_Y).
- (3 Points) Prove that if (X, d_X), (Y, d_Y), and (Z, d_Z) are metric spaces, if $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.
- (1 Point) State the definition of a closed subset.
- (3 Points) Prove that if $\mathcal{D} \subseteq Y$ is closed and $f : X \to Y$ is continuous, then $f^{-1}[\mathcal{D}] \subseteq X$ is closed.

Problem 4 (**Compactness**)

A uniformly continuous function from a metric space $(X,\, d_X)$ to a metric space $(Y,\, d_Y)$ is a function $f: X \to Y$ such that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x, x_0 \in X$, $d_X(x,\, x_0) < \delta$ implies $d_Y\big(f(x),\, f(x_0)\big) < \varepsilon$. Using cryptic notation, this says:

$$\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{x\in X}\forall_{x_0\in X}\Big(d_X(x,\, x_0) < \delta \Rightarrow d_Y\big(f(x),\, f(x_0)\big) < \varepsilon\Big) \tag{3}$$

Note, this is **stronger** than continuity. You proved in HW 1 that continuity is equivalent to:

$$\forall_{\varepsilon>0}\forall_{x\in X}\exists_{\delta>0}\forall_{x_0\in X}\Big(d_X(x,\, x_0) < \delta \Rightarrow d_Y\big(f(x),\, f(x_0)\big) < \varepsilon\Big) \tag{4}$$

The definition of uniform continuity *swaps the quantifiers*. In continuity, given an $\varepsilon > 0$ and an $x \in X$, you can find a $\delta > 0$ that may depend on ε and x, $\delta = \delta(\varepsilon,\, x)$, such that $d_X(x,\, x_0) < \delta$ implies $d_Y\big(f(x),\, f(x_0)\big) < \varepsilon$. With uniform continuity you may find a $\delta > 0$ that works for all $x \in X$, δ only depends on ε, $\delta = \delta(\varepsilon)$. The function $f(x) = \frac{1}{x}$ defined on $\mathbb{R}^+$ is an example of a function that is continuous but not uniformly continuous. Given $\varepsilon > 0$ and any $x \in \mathbb{R}^+$ you can indeed find a $\delta > 0$ such that $|x - x_0| < \delta$ implies $|\frac{1}{x} - \frac{1}{x_0}| < \varepsilon$. But as x gets smaller and smaller, closer to 0, the value of δ must get smaller too. This shows there can be no fixed positive $\delta > 0$ that works for all $x \in \mathbb{R}^+$.

In the problem you will prove the *Heine-Cantor theorem*. If $(X,\, d_X)$ is a compact metric space, if $(Y,\, d_Y)$ is a metric space, and if $f: X \to Y$ is continuous, then f is uniformly continuous.

- (2 Points) Let $\varepsilon > 0$. By continuity, for all $x \in X$, there is a $\delta_x > 0$ such that $x_0 \in X$ and $d_X(x,\, x_0) < \delta_x$ implies $d_Y\big(f(x),\, f(x_0)\big) < \varepsilon$. Let $\mathcal{U}_x = B_{\delta_x/2}^{(X,\, d_X)}(x)$ and $\mathcal{O} = \{\, \mathcal{U}_x \mid x \in X \,\}$. Show that $\mathcal{O}$ is an open cover of X.

- (2 Points) We proved that $(X,\, d_X)$ is compact if and only if every open cover $\mathcal{O}$ has a finite subcover $\Delta \subseteq \mathcal{O}$. Write $\Delta = \{\mathcal{U}_{a_0},\, \dots,\, \mathcal{U}_{a_N}\,\}$. Let $\delta = \frac{1}{2}\min(\delta_{a_0},\, \dots,\, \delta_{a_N})$. Show that if $x, x_0 \in X$ and $d_X(x,\, x_0) < \delta$, then there is an a_n such that $x, x_0 \in B_{\delta_{a_n}}^{(X,\, d)}(a_n)$ [Hint: The triangle inequality is always your friend.]

- (3 Points) Conclude that f is uniformly continuous.

Bonus: (4 Points) Prove that if $(X,\, d)$ is a compact metric space, and $f: X \to \mathbb{R}$ is continuous (with the standard metric on $\mathbb{R}$), then f is bounded. That is, there is an $M \in \mathbb{R}$ such that for all $x \in X$ we have $|f(x)| < M$.

Problem 5 (**Topological Spaces**)

You may freely use the following fact. If $f : \mathbb{R} \to \mathbb{R}$ is a non-zero polynomial, then there are only finitely many numbers $x \in \mathbb{R}$ such that $f(x) = 0$.

- (1 Point) State the definition of a topological space.
- (1 Point) State the definition of a Hausdorff topological space.
- (3 Points) Let (X, d) be a metric space and τ_d the metric topology. Prove that (X, τ_d) is a Hausdorff topological space.
- (2 Points) Let $\tau_Z \subseteq \mathcal{P}(\mathbb{R})$ be the set of all $\mathcal{U} \subseteq \mathbb{R}$ such that there is a polynomial $f : \mathbb{R} \to \mathbb{R}$ with $x \in \mathbb{R} \setminus \mathcal{U}$ if and only if $f(x) = 0$. Show that τ_Z is a topology. This is the *Zariski Topology* on $\mathbb{R}$.
- (2 Points) Show that $(\mathbb{R}, \tau_Z)$ is not a Hausdorff topological space.