Point-Set Topology: Midterm

Summer 2022

Problem 1 (**Logic**)

The truth table for a logical connective (such as $\Rightarrow$) that combines two propositions P and Q into a single proposition (like $P \Rightarrow Q$) is a table that exhausts all possibilities of P and Q being true and false. The truth table for implication is given in Tab. 1

P	Q	$P \Rightarrow Q$
False	False	True
False	True	True
True	False	False
True	True	True

Table 1: Truth Table for Implication

Prove the absorption laws. If P and Q are propositions, then P *or* (P *and* Q) if and only if P. Using $\vee$ and $\wedge$ this says:

$$P \vee (P \wedge Q) \Leftrightarrow P \tag{1}$$

Also, P *and* (P *or* Q) if and only if P. That is:

$$P \wedge (P \vee Q) \Leftrightarrow P \tag{2}$$

- (1 Point) Construct the truth table for $P \vee Q$.
- (1 Point) Construct the truth table for $P \wedge Q$.
- (1 Point) Construct the truth table for $P \vee (P \wedge Q)$.
- (1 Point) Construct the truth table for $P \wedge (P \vee Q)$.
- (1 Point) Compare these with P to prove the absorption laws.

Prove that implication can be defined by *negation* ($\neg$) and *logical or* ($\vee$).

- (1 Point) Give the truth table for $\neg P$.
- (1 Point) Give the truth table for $(\neg P) \vee Q$.
- (1 Point) Compare this with implication $P \Rightarrow Q$.

Solution.

P	Q	$P \vee Q$
False	False	False
False	True	True
True	False	True
True	True	True

Table 2: Truth Table for Logical Disjunction ($\vee$)

P	Q	$P \wedge Q$
False	False	False
False	True	False
True	False	False
True	True	True

Table 3: Truth Table for Logical Conjunction ($\wedge$)

P	Q	$P \vee Q$	$P \wedge (P \vee Q)$
False	False	False	False
False	True	True	False
True	False	True	True
True	True	True	True

Table 4: Truth Table for the First Absorption Law

P	Q	$P \wedge Q$	$P \vee (P \wedge Q)$
False	False	False	False
False	True	False	False
True	False	False	True
True	True	True	True

Table 5: Truth Table for the Second Absorption Law

In both Tab. 4 and Tab. 5 the columns for P, $P \wedge (P \vee Q)$, and $P \vee (P \wedge Q)$ are identical, meaning P is true if and only if $P \wedge (P \vee Q)$ is true, if and only if $P \vee (P \wedge Q)$ is true. This is precisely the absorption laws.

P	$\neg P$
False	True
True	False

Table 6: Truth Table for Negation

P	Q	$\neg P$	$(\neg P) \vee Q$
False	False	True	True
False	True	True	True
True	False	False	False
True	True	False	True

Table 7: Equivalent Representation of Implication

This is the same table as implication. Usually this is done the other way around. In a *Hilbert System* the main primitive notion is implication ($\Rightarrow$), and there are a few axioms for what it means. Logical or is then defined as $P \vee Q \Leftrightarrow (\neg P) \Rightarrow Q$. All of the logical symbols are further defined using implication. $\square$

Problem 2 (**Set Theory**)

Here you will construct the real numbers.

- (2 Points) Show that if A and B are sets, there is a set of all functions $f : A \to B$. [Hint: Functions are subsets $f \subseteq A \times B$ with a special property. Use the axiom of the power set and the axiom schema of specification to construct the set of all functions from A to B.]

- (1 Point) Given the rational numbers $\mathbb{Q}$ with the standard metric $d(x,\, y) = |x - y|$, state the definition of a Cauchy sequence in $\mathbb{Q}$.

- (2 Points) Let A be the set of all Cauchy sequences $a : \mathbb{N} \to \mathbb{Q}$ (This set exists by part 1 of this problem). Define the relation R on A by aRb if and only if $|a_n - b_n| \to 0$. Prove R is an equivalence relation.

- (3 Points) Let $\mathbb{R} = A/R$. Define $+$ on $\mathbb{R}$ by $[a] + [b] = [c]$ where $c : \mathbb{N} \to \mathbb{Q}$ is the sequence $c_n = a_n + b_n$. Show that $c : \mathbb{N} \to \mathbb{Q}$ is indeed a Cauchy sequence and that $+$ is well defined on $\mathbb{R}$.

Solution. By the axiom of the power set, the set $\mathcal{P}(A \times B)$ exists. Let $P(f)$ be the sentence f *is a function from* A *to* B. Let $\mathcal{F}(A,\, B)$ be defined by:

$$\mathcal{F}(A,\, B) = \{\, f \in \mathcal{P}(A \times B) \mid P(f)\,\} \tag{3}$$

Then $f \in \mathcal{F}(A,\, B)$ if and only if f is a function from A to B. That is, $\mathcal{F}(A,\, B)$ is the set of all functions $f : A \to B$. We can be very cryptic if we so desire:

$$\mathcal{F}(A,\, B) = \Big\{\, f \in \mathcal{P}(A \times B) \mid \forall_{a \in A} \exists!_{b \in B} \big((a,\, b) \in f\big)\,\Big\} \tag{4}$$

Where $\exists!$ is an extension of the $\exists$ qualifier. $\exists!$ means there exists a unique element satisfying the following proposition.

Now, using this idea to construct the real numbers. A Cauchy sequence in $\mathbb{Q}$ is a sequence $a : \mathbb{N} \to \mathbb{Q}$ such that for all $r > 0$, $r \in \mathbb{Q}$, there is an $N \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$ with $m, n > N$ we have $|a_m - a_n| < \varepsilon$. We want to say for all $r > 0$, $r \in \mathbb{R}$, but we don't have $\mathbb{R}$ yet! Now, using the first part of this problem, since a Cauchy sequence is a particular function $a : \mathbb{N} \to \mathbb{Q}$, and the set $\mathcal{F}(\mathbb{N},\, \mathbb{Q})$ of all functions from $\mathbb{N}$ to $\mathbb{Q}$ exists, we can apply the sentence $P(a)$, a *is a Cauchy sequence* to the set $\mathcal{F}(\mathbb{N},\, \mathbb{Q})$ and obtain the set A of all Cauchy sequences in $\mathbb{Q}$. The relation R on A defined by aRb if and only if $|a_n - b_n| \to 0$ is an equivalence relation. It is reflexive since $|a_n - a_n| = 0$ for all $n \in \mathbb{N}$, so indeed $|a_n - a_n| \to 0$. It is symmetric. If aRb, then:

$$\lim_{n\to\infty} |b_n - a_n| = \lim_{n\to\infty} |(-1)(a_n - b_n)| = \lim_{n\to\infty} |a_n - b_n| = 0 \tag{5}$$

and hence bRa. Lastly, it is transitive. If aRb and bRc, then $|a_n - b_n| \to 0$ and $|b_n - c_n| \to 0$. Let $\varepsilon > 0$. There exists $N_0, N_1 \in \mathbb{N}$ such that $n \in \mathbb{N}$ and $n > N_0$

implies $|a_n - b_n| < \varepsilon/2$, and $n \in \mathbb{N}$ and $n > N_1$ implies $|b_n - c_n| < \varepsilon/2$. Let $N = \max(N_0, N_1)$. Then for $n \in \mathbb{N}$ and $n > N$ we have:

$$|a_n - c_n| = |a_n - b_n + b_n - c_n| \leq |a_n - b_n| + |b_n - c_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \tag{6}$$

and hence $|a_n - c_n| \to 0$. So R is reflexive, symmetric, and transitive, and is therefore an equivalence relation.

Defining $[a] + [b] = [c]$ where $c : \mathbb{N} \to \mathbb{Q}$ is the sequence $c_n = a_n + b_n$, first c is indeed a Cauchy sequence. Let $\varepsilon > 0$. Since a is a Cauchy sequence there is an $N_0 \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$ with $m, n > N_0$ we have $|a_m - a_n| < \varepsilon/2$. But b is a Cauchy sequence, so there is an $N_1 \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$ with $m, n > N_1$ we have $|b_m - b_n| < \varepsilon/2$. Let $N = \max(N_0, N_1)$. Then for all $m, n \in \mathbb{N}$ with $m, n > N$ we have:

$$\begin{aligned}
|c_m - c_n| &= |(a_m + b_m) - (a_n + b_n)| && (7)\\
&= |a_m + b_m - a_n - b_n| && (8)\\
&= |(a_m - a_n) + (b_m - b_n)| && (9)\\
&\leq |a_m - a_n| + |b_m - b_n| && (10)\\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && (11)\\
&= \varepsilon && (12)
\end{aligned}$$

so c is a Cauchy sequence. Addition is well-defined. If $a, b, x, y \in A$ are Cauchy sequences, and if $[a] = [x]$ and $[b] = [y]$, then:

$$\begin{aligned}
\lim_{n\to\infty} |(a_n + b_n) - (x_n + y_n)| &= \lim_{n\to\infty} |a_n + b_n - x_n - y_n| && (13)\\
&= \lim_{n\to\infty} |(a_n - x_n) + (b_n - y_n)| && (14)\\
&\leq \lim_{n\to\infty} \big(|a_n - x_n| + |b_n - y_n|\big) && (15)\\
&= \lim_{n\to\infty} |a_n - x_n| + \lim_{n\to\infty} |b_n - y_n| && (16)\\
&= 0 + 0 && (17)\\
&= 0 && (18)
\end{aligned}$$

And hence $(a + b)R(x + y)$, meaning $[a] + [b] = [x] + [y]$, so addition is well-defined. $\square$

Problem 3 (**Metric Spaces**)

- (1 Point) State the definition of a metric space.
- (1 Point) State the definition of a convergent sequence.
- (1 Point) State the definition of a continuous function from a metric space $(X,\, d_X)$ to a metric space $(Y,\, d_Y)$.
- (3 Points) Prove that if $(X,\, d_X)$, $(Y,\, d_Y)$, and $(Z,\, d_Z)$ are metric spaces, if $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.
- (1 Point) State the definition of a closed subset.
- (3 Points) Prove that if $\mathcal{D} \subseteq Y$ is closed and $f : X \to Y$ is continuous, then $f^{-1}[\mathcal{D}] \subseteq X$ is closed.

Solution. A metric space is a set X with a metric function $d : X \times X \to \mathbb{R}$ that satisfies the following:

- Positive-Definiteness

  $d(x,\, y) \geq 0$ for all $x, y \in X$ and $d(x,\, y) = 0$ if and only if $x = y$.
- Symmetry

  $d(x,\, y) = d(y,\, x)$ for all $x, y \in X$.
- Triange Inequality

  $d(x,\, z) \leq d(x,\, y) + d(y,\, z)$ for all $x, y, z \in X$.

A convergent sequence in a metric space $(X,\, d)$ is a sequence $a : \mathbb{N} \to X$ such that there is an $x \in X$ such that for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $n \in \mathbb{N}$ and $n > N$ implies $d(x,\, a_n) < \varepsilon$.

A continuous function between metric spaces is a function that maps convergent sequences to convergent sequences. That is, given metric spaces $(X,\, d_X)$ and $(Y,\, d_Y)$, a continuous function is a function $f : X \to Y$ such that for all convergent sequences $a : \mathbb{N} \to X$ such that $a_n \to x$ for some $x \in X$, it is true that $f(a_n) \to f(x)$.

Suppose $f : X \to Y$ and $g : Y \to Z$ are continuous. Let $a : \mathbb{N} \to X$ be a convergent sequence with $a_n \to x \in X$. Since f is continuous, $f(a_n) \to f(x)$. Since g is continuous and $f(a)$ is a convergent sequence, $g\big(f(a_n)\big) \to g\big(f(x)\big)$. But $(g \circ f)(a_n) = g\big(f(a_n)\big)$, so $(g \circ f)(a_n) \to (g \circ f)(x)$. Hence, $g \circ f$ is continuous.

A closed set in a metric space $(X,\, d)$ is a subset $\mathcal{C} \subseteq X$ such that for every sequence $a : \mathbb{N} \to \mathcal{C}$ such that $a_n \to x \in X$ it is true that $x \in \mathcal{C}$. That is, $\mathcal{C}$ has all of its limit points.

Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ continuous. Let $\mathcal{D} \subseteq Y$ be closed and $\mathcal{C} = f^{-1}[\mathcal{D}]$. Let $a : \mathbb{N} \to \mathcal{C}$ be a convergent sequence with limit $x \in X$. Since f is continuous, $f(a_n) \to f(x)$. But $\mathcal{D}$ is closed and $f(a_n) \in \mathcal{D}$, so $f(x) \in \mathcal{D}$. Hence $x \in \mathcal{C}$, so $\mathcal{C}$ is closed. $\square$

Problem 4 (**Compactness**)

A uniformly continuous function from a metric space $(X,\, d_X)$ to a metric space $(Y,\, d_Y)$ is a function $f : X \to Y$ such that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x, x_0 \in X$, $d_X(x,\, x_0) < \delta$ implies $d_Y\big(f(x),\, f(x_0)\big) < \varepsilon$. Using cryptic notation, this says:

$$\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{x\in X}\forall_{x_0\in X}\Big(d_X(x,\, x_0) < \delta \Rightarrow d_Y\big(f(x),\, f(x_0)\big) < \varepsilon\Big) \tag{19}$$

Note, this is **stronger** than continuity. You proved in HW 1 that continuity is equivalent to:

$$\forall_{\varepsilon>0}\forall_{x\in X}\exists_{\delta>0}\forall_{x_0\in X}\Big(d_X(x,\, x_0) < \delta \Rightarrow d_Y\big(f(x),\, f(x_0)\big) < \varepsilon\Big) \tag{20}$$

The definition of uniform continuity *swaps the quantifiers*. In continuity, given an $\varepsilon > 0$ and an $x \in X$, you can find a $\delta > 0$ that may depend on ε and x, $\delta = \delta(\varepsilon,\, x)$, such that $d_X(x,\, x_0) < \delta$ implies $d_Y\big(f(x),\, f(x_0)\big) < \varepsilon$. With uniform continuity you may find a $\delta > 0$ that works for all $x \in X$, δ only depends on ε, $\delta = \delta(\varepsilon)$. The function $f(x) = \frac{1}{x}$ defined on $\mathbb{R}^+$ is an example of a function that is continuous but not uniformly continuous. Given $\varepsilon > 0$ and any $x \in \mathbb{R}^+$ you can indeed find a $\delta > 0$ such that $|x - x_0| < \delta$ implies $|\frac{1}{x} - \frac{1}{x_0}| < \varepsilon$. But as x gets smaller and smaller, closer to 0, the value of δ must get smaller too. This shows there can be no fixed positive $\delta > 0$ that works for all $x \in \mathbb{R}^+$.

In the problem you will prove the *Heine-Cantor theorem*. If $(X,\, d_X)$ is a compact metric space, if $(Y,\, d_Y)$ is a metric space, and if $f : X \to Y$ is continuous, then f is uniformly continuous.

- (2 Points) Let $\varepsilon > 0$. By continuity, for all $x \in X$, there is a $\delta_x > 0$ such that $x_0 \in X$ and $d_X(x,\, x_0) < \delta_x$ implies $d_Y\big(f(x),\, f(x_0)\big) < \varepsilon$. Let $\mathcal{U}_x = B_{\delta_x/2}^{(X,\, d_X)}(x)$ and $\mathcal{O} = \{\,\mathcal{U}_x \mid x \in X\,\}$. Show that $\mathcal{O}$ is an open cover of X.
- (2 Points) We proved that $(X,\, d_X)$ is compact if and only if every open cover $\mathcal{O}$ has a finite subcover $\Delta \subseteq \mathcal{O}$. Write $\Delta = \{\,\mathcal{U}_{a_0},\, \dots,\, \mathcal{U}_{a_N}\,\}$. Let $\delta = \frac{1}{2}\min(\delta_{a_0},\, \dots,\, \delta_{a_N})$. Show that if $x, x_0 \in X$ and $d_X(x,\, x_0) < \delta$, then there is an a_n such that $x, x_0 \in B_{\delta_{a_n}}^{(X,\, d)}(a_n)$ [Hint: The triangle inequality is always your friend.]
- (3 Points) Conclude that f is uniformly continuous.

Bonus: (4 Points) Prove that if $(X,\, d)$ is a compact metric space, and $f : X \to \mathbb{R}$ is continuous (with the standard metric on $\mathbb{R}$), then f is bounded. That is, there is an $M \in \mathbb{R}$ such that for all $x \in X$ we have $|f(x)| < M$.

Solution. For all $\mathcal{U} \in \mathcal{O}$, $\mathcal{U}$ is an open ball, and hence open. But moreover, for all $x \in X$, since δ_x is chosen to be positive, we have that $B_{\delta_x}^{(X,\,d)}(x)$ is non-empty since it contains the point x. Since this is true of all $x \in X$, $\mathcal{O}$ is a collection of open sets that cover X, and is therefore an open cover.

Let $x, x_0 \in X$ be such that $d_X(x,\, x_0) < \delta$. Since Δ is a cover of X there is a $\mathcal{U}_n \in \Delta$ such that $x \in \mathcal{U}_n$. But $\mathcal{U}_n = B_{\delta_{a_n}/2}^{(X,\,d)}(a_n)$, so $d_X(x,\, a_n) < \delta_{a_n}/2$. But then, by the triangle inequality, we have:

$$d_X(x_0,\, a_n) \leq d_X(x_0,\, x) + d_X(x,\, a_n) < \delta + \frac{\delta_{a_n}}{2} \leq \frac{\delta_{a_n}}{2} + \frac{\delta_{a_n}}{2} = \delta_{a_n} \tag{21}$$

by the definition of δ. So $x, x_0 \in B_{\delta_{a_n}}^{(X,\,d)}(a_n)$.

Let $x, x_0 \in X$ with $d_X(x,\, x_0) < \delta$. Then there is an a_n such that $x, x_0 \in B_{\delta_{a_n}}^{(X,\,d)}(a_n)$. But then:

$$d_Y\big(f(x),\, f(x_0)\big) \leq d_Y\big(f(x),\, f(a_n)\big) + d_Y\big(f(x_0),\, f(a_n)\big) < \varepsilon + \varepsilon = 2\varepsilon \tag{22}$$

Since 2ε can be made arbitrarily small, and since δ was chosen independent of x, f is uniformly continuous.

For the bonus, suppose f is not bounded. Then for all $M \in \mathbb{R}$ there is an $x \in X$ such that $|f(x)| \geq M$. In particular, for all $n \in \mathbb{N}$ there is an $a_n \in X$ such that $|f(a_n)| \geq n$. But then $a : \mathbb{N} \to X$ is a sequence in a compact metric space, so there is a convergent subsequence a_k. Let $x \in X$ be the limit, $a_{k_n} \to x$. But f is continuous, so if $a_{k_n} \to x$, then $f(a_{k_n}) \to f(x)$. Let $N \in \mathbb{N}$ be such that $N > |f(x)| + 1$. But then for all $n \in \mathbb{N}$ with $n > N$ we have $|f(x) - f(a_{k_n})| > 1$, so $f(a_{k_n})$ can't converge to $f(x)$, a contradiction. So f is bounded. □

Problem 5 (Topological Spaces)

You may freely use the following fact. If $f : \mathbb{R} \to \mathbb{R}$ is a non-zero polynomial, then there are only finitely many numbers $x \in \mathbb{R}$ such that $f(x) = 0$.

- (1 Point) State the definition of a topological space.
- (1 Point) State the definition of a Hausdorff topological space.
- (3 Points) Let (X, d) be a metric space and τ_d the metric topology. Prove that (X, τ_d) is a Hausdorff topological space.
- (2 Points) Let $\tau_Z \subseteq \mathcal{P}(\mathbb{R})$ be the set of all $\mathcal{U} \subseteq \mathbb{R}$ such that there is a polynomial $f : \mathbb{R} \to \mathbb{R}$ with $x \in \mathbb{R} \setminus \mathcal{U}$ if and only if $f(x) = 0$. Show that τ_Z is a topology. This is the *Zariski Topology* on $\mathbb{R}$.
- (2 Points) Show that $(\mathbb{R}, \tau_Z)$ is not a Hausdorff topological space.

Solution. A topological space is a set X with a topology τ, which is a subset $\tau \subseteq \mathcal{P}(X)$ satisfying:

- $\emptyset \in \tau$
- $X \in \tau$
- If $\mathcal{U}, \mathcal{V} \in \tau$, then $\mathcal{U} \cap \mathcal{V} \in \tau$.
- If $\mathcal{O} \subseteq \tau$, then $\bigcup \mathcal{O} \in \tau$.

A Hausdorff topological space is a topological space (X, τ) such that for all $x, y \in X$ with $x \neq y$ there are open sets $\mathcal{U}, \mathcal{V} \in \tau$ such that $x \in \mathcal{U}$, $y \in \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V} = \emptyset$.

A metrizable space is Hausdorff. Let (X, τ) be metrizable, with metric d inducing the topology τ. Let $x, y \in X$ be distinct, $x \neq y$. Since d is a metric, $d(x, y) > 0$. Let $\varepsilon = \frac{1}{2}d(x, y)$. Let $\mathcal{U} = B_{\varepsilon}^{(X,\,d)}(x)$ and $\mathcal{V} = B_{\varepsilon}^{(X,\,d)}(y)$. Then, since open balls are open, $\mathcal{U}$ and $\mathcal{V}$ are elements of τ. Suppose $z \in \mathcal{U} \cap \mathcal{V}$. Then:

$$d(x, y) \leq d(x, z) + d(z, y) < \varepsilon + \varepsilon = d(x, y) \tag{23}$$

so $d(x, y) < d(x, y)$, a contradiction, and therefore $\mathcal{U} \cap \mathcal{V} = \emptyset$. That is, (X, τ) is Hausdorff.

The Zariski topology is a topology. The entire set is in it since $f(x) = 1$ is a polynomial and $f(x) = 0$ if and only if $x \in \emptyset$. So $\mathbb{R} = \mathbb{R} \setminus \emptyset$ is an element of τ_Z. Similarly, $f(x) = 0$ is a polynomial and $f(x) = 0$ for all $x \in \mathbb{R}$, hence $\emptyset = \mathbb{R} \setminus \mathbb{R}$ is in τ_Z. Let $\mathcal{U}, \mathcal{V} \in \tau_Z$ be open sets. Then there are polynomials f and g corresponding to $\mathcal{U}$ and $\mathcal{V}$, respectively. But the product of polynomials is a polynomial, so $h = fg$ is a polynomial. But then $h(x) = 0$ if and only if

$f(x)g(x) = 0$. But $f(x)g(x) = 0$ if and only if $f(x) = 0$ or $g(x) = 0$ (Euclid's theorem). But then $h(x) = 0$ if and only if $x \in \mathbb{R} \setminus \mathcal{U}$ or $x \in \mathbb{R} \setminus \mathcal{V}$. Thus $h(x) = 0$ if and only if $x \in \mathbb{R} \setminus (\mathcal{U} \cap \mathcal{V})$, so $\mathcal{U} \cap \mathcal{V}$ is open. Lastly, let $\mathcal{O} \subseteq \tau_Z$. If $\mathcal{O}$ is empty, the union is empty, and the empty set is an element of τ_Z. If $\mathbb{R} \in \mathcal{O}$, then $\bigcup \mathcal{O} = \mathbb{R}$, and $\mathbb{R} \in \tau_Z$. So suppose $\mathcal{O}$ is non-empty and $\mathbb{R} \notin \mathcal{O}$. But then every $\mathcal{U} \in \mathcal{O}$ corresponds to a polynomial f where $f(x) = 0$ for at least some $x \in \mathbb{R}$. Let $\mathcal{U} \in \mathcal{O}$ and let f be the corresponding polynomial. But then $\mathbb{R} \setminus \mathcal{U}$ is finite since a non-zero polynomial has only finitely many zeros. But then:

$$\mathbb{R} \setminus \bigcup \mathcal{O} \subseteq \mathbb{R} \setminus \mathcal{U} \tag{24}$$

So $\mathbb{R} \setminus \bigcup \mathcal{O}$ is finite. Let the elements be $x_0, \dots, x_n$. Let $h(x)$ be defined by:

$$h(x) = \prod_{k=0}^{n} (x - x_k) = (x - x_0)(x - x_1) \cdots (x - x_n) \tag{25}$$

Then $h(x) = 0$ if and only if $x \in \mathbb{R} \setminus \bigcup \mathcal{O}$. Hence $\bigcup \mathcal{O}$ is open.

$(\mathbb{R}, \tau_Z)$ is not Hausdorff. Let $\mathcal{U}, \mathcal{V}$ be non-empty proper open subsets. Then, since non-zero polynomials have only finitely many zeros, $\mathbb{R} \setminus \mathcal{U}$ and $\mathbb{R} \setminus \mathcal{V}$ are finite. But then $\mathcal{U} \cap \mathcal{V}$ must be infinite since $\mathbb{R}$ is infinite, and hence (X, τ_Z) can not be Hausdorff. □