Point-Set Topology: Homework 1

Summer 2022

Problem 1 This problem explores the use of the word *or* in mathematics. You will prove the *principle of explosion*. The principle says that if P is a statement that is both true and false, then for any statement Q, Q is true. A system that contains a sentence that is both true and false is called *inconsistent*. The principle of explosion shows that inconsistent systems are very boring.

- (2 Points) Let P be a statement that is true, and let Q be any other claim. Since P is true, what can you conclude about P *or* Q?
- (2 Points) Suppose P is also false. Using what you've concluded about P *or* Q, what can you prove about Q?

Problem 2 Here you will explore more set theory. You will prove the *axiom of unrestricted comprehension* is inconsistent. The axiom allows you to construct a set arbitrarily using any sentence. That is, if $P(x)$ is a sentence, you may collect all x such that $P(x)$ is true. You could write:

$$A = \{\, x \mid P(x) \,\}$$

- (1 Point) Let $P(x)$ be the sentence *x is a set*. Let A be the set $A = \{\, x \mid P(x) \,\}$. That is:
$$A = \{\, x \mid x \text{ is a set} \,\}$$
Describe the set A. Is it true that $A \in A$?
- (2 Points) Let B be the set
$$B = \{\, x \in A \mid x \notin x \,\}$$
Prove that $B \in B$. (Hint: Suppose $B \notin B$ and arrive at a contradiction).
- (2 Points) Now prove that $B \notin B$. (Hint: Same as before. Suppose $B \in B$ and arrive at a contradiction).
- (1 Point) Using the previous problem, why should we not accept the axiom of unrestricted comprehension as true?

Problem 3 The axiom of infinity tells us $\mathbb{N} = \{\, 0, 1, 2, \dots \,\}$ exists. It does not tell us $\mathbb{Z} = \{\, \dots, -2, -1, 0, 1, 2, \dots \,\}$ exists, but we can construct it. For this problem we can assume addition $(+)$ for natural numbers exists.

- (2 Points) Consider the set $\mathbb{N}\times\mathbb{N}$. Define R to be the relation $(a,b)R(c,d)$ if and only if $a+d=b+c$. Prove this is an equivalence relation.
- (1 Point) Describe the equivalence class of (a,b) geometrically. (Hint: $\mathbb{N}\times\mathbb{N}$ is a lattice of points in the plane. Describe the equivalence class of (a,b) using this lattice).
- (2 Points) For equivalence classes $[(a,b)]$ and $[(c,d)]$, define $[(a,b)]+[(c,d)]=[(a+c,b+d)]$. Prove this is well-defined.
- (1 Point) With this construction we now write (for convenience) $0=[(0,0)]$, $n=[(n,0)]$, and $-n=[(0,n)]$. Justifiy this notation (Does $[(0,0)]$ behave like 0? Does $[(0,n)]$ act like $-n$?)

Problem 4 Now that we have constructed $\mathbb{Z}$, let's construct $\mathbb{Q}$, the set of rational numbers. Consider the set $\mathbb{Z}\times(\mathbb{Z}\setminus\{0\})$. Define the relation R by $(a,b)R(c,d)$ if and only if $ad=bc$. (We are assuming we have already constructed multiplication for integers $n\in\mathbb{Z}$).

- (2 Points) Prove R is an equivalence relation.
- (2 Points) For equivalence clases $[(a,b)]$ and $[(c,d)]$, define $[(a,b)]+[(c,d)]=[(ad+bc,bd)]$ (This is *cross-multiplying*). Prove this is well-defined.

With this we write $[(a,b)]=\frac{a}{b}$.

Problem 5 Prove Cantor's theorem. If A is a set, and $\mathcal{P}(A)$ is the power set of A, then there is no surjection $f:A\to\mathcal{P}(A)$.

- (1 Point) Suppose there is a surjection $f:A\to\mathcal{P}(A)$. Consider the set $B=\{x\in A\mid x\notin f(x)\}$. Describe in words what the set B contains.
- (2 Points) Since $B\subseteq A$, and since f is surjective, there is an element $a\in A$ such that $f(a)=B$. Show that this is a contradiction.
- (1 Point) Construct an injective function $g:A\to\mathcal{P}(A)$. (Hint: Given $a\in A$, what's an "obvious" subset we can send a to?)

Problem 6 We proved in class that a function $f:X\to Y$ from a metric space (X,d_X) to a metric space (Y,d_Y) is continuous if and only if for every open subset $\mathcal{V}\subseteq Y$, the pre-image $f^{-1}[\mathcal{V}]$ is also open. You will now prove the equivalence of the third definition of continuity.

- (3 Points) Prove that if f is continuous, then for all $\varepsilon>0$, and for all $x\in X$, there is a $\delta>0$ such that if $x_0\in X$ and $d_X(x,x_0)<\delta$, then $d_Y\big(f(x),f(x_0)\big)<\varepsilon$ (Hint: Suppose not. Then there is an $\varepsilon>0$ such that for each $n\in\mathbb{N}$, $n>0$, there is a point $a_n\in X$ with $d_X(x,a_n)<\frac{1}{n}$ and $d_Y\big(f(x),f(a_n)\big)\geq\varepsilon$. What is $\lim_{n\to\infty}a_n$? What is $\lim_{n\to\infty}f(a_n)$? Is there a contradiction?)

- (3 Points) Prove that if $f : X \to Y$ is a function such that for all $\varepsilon > 0$ and for all $x \in X$, there is a $\delta > 0$ such that for all $x_0 \in X$, $d_X(x, x_0) < \delta$ implies $d_Y\big(f(x), f(x_0)\big) < \varepsilon$, then f is continuous. (Hint: Let $a_n \to x$ by a convergent sequence. What does this property say about $\lim_{n\to\infty} f(a_n)$?)

Problem 7 A locally compact metric space is a metric space (X, d) where for all $x \in X$ there is a compact set K and an open set $\mathcal{U}$ such that $x \in \mathcal{U}$ and $\mathcal{U} \subseteq K$ (See Fig. 1).

- (2 Points) Construct a metric space that is *not* locally compact. Explain why it is not locally compact. (Hint: Il est utile de penser á la France).

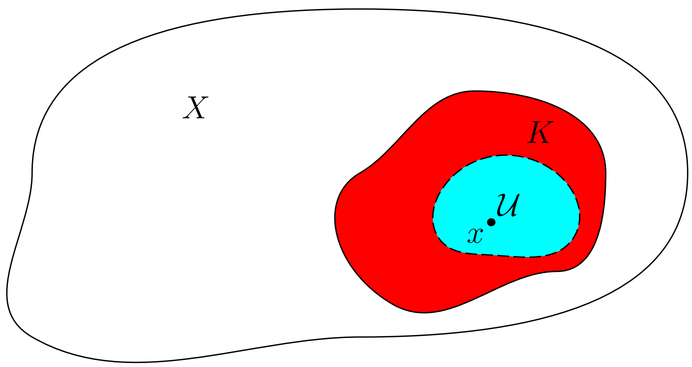

Figure 1: Diagram for Locally Compact Metric Spaces