Point-Set Topology: Homework 2

Summer 2022

Problem 1 A few more notes about metric spaces. A *contraction* on a metric space $(X,\, d)$ is a function $f : X \to X$ such that for all $x, y \in X$ it is true that $d\big(f(x),\, f(y)\big) \leq r\, d(x,\, y)$ for some fixed $0 \leq r < 1$. This means the function f *squeezes* the points together. You will prove one of the most celebrated theorems of the theory of metric spaces, the *Banach Fixed Point Theorem.* If $(X,\, d)$ is a non-empty complete metric space, and if $f : X \to X$ is a contraction, then there is a unique point $x \in X$ such that $f(x) = x$. That is, f has a unique *fixed-point*, a point that is not changed by f.

- (2 Points) Prove that a contraction $f : X \to X$ is continuous.
- (2 Points) Prove that if $f : X \to X$ has a fixed-point $x \in X$, then x is the only fixed-point. [Hint: What if $y \in X$ is another fixed-point? Anything wrong?]
- (2 Points) Let $a_0 \in X$ be arbitrary, define a_n inductively via $a_{n+1} = f(a_n)$. Prove that for all $n \in \mathbb{N}$, $d(a_{n+1},\, a_n) \leq r^n d(a_1,\, a_0)$, where $0 \leq r < 1$ is a value such that for all $x, y \in X$ we have $d\big(f(x),\, f(y)\big) \leq r\, d(x,\, y)$.
- (2 Points) Conclude that $a : \mathbb{N} \to X$ is a Cauchy sequence. [Hint: Apply the triangle inequality and use the geometric series from calculus].
- (2 Points) Since $(X,\, d)$ is complete, the sequence converges. Let $x \in X$ be such that $a_n \to x$. Show that $f(x) = x$. [Hint: Use the continuity of f that you proved in the first part of this problem]

The first application of this is the *Picard-Lindelöf theorem*, a theorem with widespread use in analysis, geometry, and physics. It says if $f(t,\, \mathbf{x})$ is a *nice* function (continuous in t, Lipschitz continuous in $\mathbf{x}$) from some closed rectangle R in $\mathbb{R} \times \mathbb{R}^n$, if $(t_0,\, \mathbf{x}_0) \in R$, then there is an $\varepsilon > 0$ and a unique function $\mathbf{x}(t)$ such that:

$$\mathbf{x}'(t) = f\big(t,\, \mathbf{x}(t)\big) \tag{1}$$

satisfying the initial value problem $\mathbf{x}(t_0) = \mathbf{x}_0$ in the interval $(t_0 - \varepsilon,\, t_0 + \varepsilon)$. In the single variable case, this implies we may solve $\dot{x}(t) = f\big(t,\, x(t)\big)$ for smooth functions f. The proof constructs the unique solution. Define $\phi_0(t) = t_0$.

Inductively define $\phi_n(t)$ via:

$$\phi_{n+1}(t) = y_0 + \int_{t_0}^{t} f\big(s,\,\phi_n(s)\big)\,\mathrm{d}s \tag{2}$$

The Banach fixed-point theorem shows, with a bit of work, that $\phi_n(t)$ converges to a function and that this limit function satisfies the initial value problem.

Problem 2 A dense subset of a topological space $(X,\,\tau)$ is a subset $A \subseteq X$ such that $\mathrm{Cl}_\tau(A) = X$. That is, every point in X is a limit point of A. For example, the rationals $\mathbb{Q}$ are a dense subset of the reals $\mathbb{R}$. A Baire topological space is a topological space $(X,\,\tau)$ such that for any non-empty countable set $\mathcal{O} \subseteq \tau$ with the property that $\mathcal{U} \in \mathcal{O}$ implies $\mathcal{U}$ is dense, the intersection $\bigcap \mathcal{O}$ is also dense. Here you will prove the first of Baire's Category Theorems (Note: The Baire category theorem has absolutely nothing to do with category theory. The terminology for this theorem came long before category theory was initiated). If $(X,\,d)$ is a complete metric space, and if τ_d is the metric topology, then $(X,\,\tau_d)$ is a Baire topological space.

- (2 Points) Prove that, for a topological space $(Y,\,\tau_Y)$, $A \subseteq Y$ is dense if and only if for every non-empty open set $\mathcal{U} \subseteq Y$, the intersection $\mathcal{U} \cap A$ is non-empty.

- (2 Points) It now suffices to prove that if $\mathcal{W} \subseteq X$ is open and non-empty, then $\mathcal{W} \cap \bigcap \mathcal{O}$ is non-empty. Show that if $\mathcal{V}$ is an open ball, $\mathcal{V} = B_r^{(X,\,d)}(x)$, then there is an $\varepsilon > 0$ such that $\mathrm{Cl}_\tau\big(B_\varepsilon^{(X,\,d)}(x)\big) \subseteq \mathcal{V}$. That is, there is always a *closed ball* inside of an open ball.

- (2 Points) Since $\mathcal{O}$ is countable, there is a surjective sequence $\mathcal{U} : \mathbb{N} \to \mathcal{O}$. That is, we may list the elements of $\mathcal{O}$ as $\mathcal{U}_0$, $\mathcal{U}_1$, and so on. Since $\mathcal{U}_0$ is open and dense, $\mathcal{U}_0 \cap \mathcal{W}$ is non-empty. Hence there an $a_0 \in \mathcal{U}_0 \cap \mathcal{W}$. Since the intersection of open sets is open, there is a positive $r_0 < 1$ such that $B_{r_0}^{(X,\,d)}(a_0) \subseteq \mathcal{U}_0 \cap \mathcal{W}$. By the previous part of the problem, there is a positive $\varepsilon_0 < r_0$ such that $\mathrm{Cl}_{\tau_d}\big(B_{\varepsilon_0}^{(X,\,d)}(a_0)\big) \subseteq B_{r_0}^{(X,\,d)}(a_0)$. Recursively we may define a_n, r_n, and ε_n such that $r_n < \frac{1}{n+1}$, and:

  $$\mathrm{Cl}_{\tau_d}\big(B_{\varepsilon_n}^{(X,\,d)}(a_n)\big) \subseteq B_{r_n}^{(X,\,d)}(a_n) \subseteq \mathcal{W} \cap \bigcap_{k=0}^{n} \mathcal{U}_n \tag{3}$$

  and such that:

  $$\mathrm{Cl}_{\tau_d}\big(B_{\varepsilon_{n+1}}^{(X,\,d)}(a_{n+1})\big) \subseteq B_{\varepsilon_n}^{(X,\,d)}(a_n) \tag{4}$$

  Show that $a : \mathbb{N} \to X$ is a Cauchy sequence.

- (2 Points) Since $(X,\,d)$ is complete, there is an $x \in X$ such that $a_n \to x$. Show that for all $n \in \mathbb{N}$ it is true that $x \in \mathcal{U}_n$. [Hint: Since $\mathrm{Cl}_{\tau_d}\big(B_{\varepsilon_n}^{(X,\,d)}(a_n)\big)$ is closed, it contains all of its limit points. Show that x is a limit point of this for all n. Conclude that x is in $\mathcal{U}_n$ since $\mathrm{Cl}_{\tau_d}\big(B_{\varepsilon_n}^{(X,\,d)}(a_n)\big) \subseteq \mathcal{U}_n$.

- (2 Points) Show that $x \in \mathcal{W}$ as well, and therefore $x \in \mathcal{W} \cap \bigcap \mathcal{O}$, proving the intersection is non-empty, and therefore $\bigcap \mathcal{O}$ is dense.

Problem 3 From class, a Kolmogorov topology on a set X is a topology τ on X such that for all $x, y \in X$, there is an open set $\mathcal{U} \in \tau$ such that either $x \in \mathcal{U}$ and $y \notin \mathcal{U}$, or $x \notin \mathcal{U}$ and $y \in \mathcal{U}$. That is, a Kolmogorov topology is a topology where it is always possible to tell two points apart using open sets.

- (2 Points) There are 8,977,053,873,043 distinct topologies on the set $\mathbb{Z}_{10}$, 6,611,065,248,783 Kolmogorov topologies, and 4,717,687 topologies that are not homeomorphic. Quite a lot. It would be cruel to ask you to find them all. Instead, find all distinct topologies on $\mathbb{Z}_2$ (there are 4), all distinct Kolmogorov topologies (there's 3), all non-homeomorphic topologies (3), all non-homeomorphic Kolmogorov topologies (2), and all Hausdorff topologies (1). [Hint: This may seem like a lot, but it really isn't. Find the 4 topologies on $\mathbb{Z}_2$. Then examine which are Kolmogorov and which are homeomorphic, etc.]

- (2 Points) On $\mathbb{Z}_3$ there are 29 distinct topologies, 19 distinct Kolmogorov topologies, 9 non-homeomorphic topologies, and 5 non-homeomorphic Kolmogorov topologies. Find 2 non-homeomorphic Kolmogorov topologies. [Hint: Hausdorff implies Kolmogorov. Can you find the Hausdorff topology?]

Problem 4 (4 Points) Let (X, τ) be a sequential topological space and R an equivalence relation on X. Prove that the quotient space $(X/R, \tau_{X/R})$ is sequential as well.

Problem 5 Kazimeirz Kuratowski gave an alternative, but equivalent, definition of topology. To him the notion of *closure* was sufficient to describe topological spaces. A Kuratowski closure operator on a set X is a function $\sigma : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ such that, for all $A, B \subseteq X$:

1. $\sigma(\emptyset) = \emptyset$
2. $A \subseteq \sigma(A)$
3. $\sigma(A) = \sigma\big(\sigma(A)\big)$
4. $\sigma(A \cup B) = \sigma(A) \cup \sigma(B)$

A Kuratowski space is an ordered pair (X, σ) where X is a set and σ is a Kuratowski closure operator on X. We have seen in class that, if (X, τ) is a topological space, then Cl_τ is a Kuratowski closure operator. Now, let's go the other way.

- (2 Points) Show that, given (X, σ), the set τ_σ defined by:

$$\tau_\sigma = \{\, X \setminus \mathcal{C} \in \mathcal{P}(X) \mid \sigma(\mathcal{C}) = \mathcal{C} \,\} \tag{5}$$

  is a topology on X. (We proved that, in topological spaces, $A \subseteq X$ being closed is equivalent to $\mathrm{Cl}_\tau(A) = A$. We are intuitively defining τ_σ as the set of all *complements of closed sets*).

- (6 Points) If (X, σ_X) and (Y, σ_Y) are Kuratowski spaces, $f : X \to Y$ is continuous if for all $A \subseteq X$ it is true that $f[\sigma_X(A)] \subseteq \sigma_Y(f[A])$. Show this is equivalent to continuity in topology. That is, if (X, τ_X) and (Y, τ_Y) are topological spaces, then $f : X \to Y$ is continuous if and only if for all $A \subseteq X$ it is true that $f[\mathrm{Cl}_{\tau_X}(A)] \subseteq \mathrm{Cl}_{\tau_Y}(f[A])$. [Hint: We proved $f : X \to Y$ is continuous if and only if for all closed $\mathcal{D} \subseteq Y$, the pre-image $f^{-1}[\mathcal{D}]$ is closed. Use this definition.]