Point-Set Topology: Homework 3

Summer 2022

Problem 1 From class, given a collection of topological spaces (X_α, τ_α) (for all α in some indexing set I), the product topology and box topology are formed as follows. For the box topology τ_{Box} we take as a basis:

$$\mathcal{B}_{\text{Box}} = \left\{ \prod_{\alpha \in I} \mathcal{U}_\alpha \mid \mathcal{U}_\alpha \in \tau_\alpha \text{ for all } \alpha \in I \right\} \tag{1}$$

For the product topology τ_{Prod}, we take as a basis:

$$\mathcal{B}_{\text{Prod}} = \left\{ \prod_{\alpha \in I} \mathcal{U}_\alpha \mid \mathcal{U}_\alpha \in \tau_\alpha, \mathcal{U}_\alpha = X_\alpha \text{ for all but finitely many } \alpha \in I \right\} \tag{2}$$

from this definition it is hopefully clear that $\tau_{\text{Prod}} \subseteq \tau_{\text{Box}}$. Because of this, given any topological space (Y, τ_Y), and a function $f : Y \to \prod_{\alpha \in I} X_\alpha$, if f is continuous with respect to the box topology, then it is continuous with respect to the product topology. You will prove the converse is **false**.

- (3 Points) Prove that if (X, τ_X) and (Y, τ_Y) are topological spaces, then $f : X \to Y$ is continuous if and only if for every $x \in X$ and for every $\mathcal{V} \in \tau_Y$ such that $f(x) \in \mathcal{V}$, there is a $\mathcal{U} \in \tau_X$ such that $x \in \mathcal{U}$ and $f[\mathcal{U}] \subseteq \mathcal{V}$. That is, continuity can be described by forward images, as well as by pre-images.

- (2 Points) Let $\mathbb{R}^\infty = \prod_{n \in \mathbb{N}} \mathbb{R}$. This is, as a set, the set of all sequences $a : \mathbb{N} \to \mathbb{R}$ (review what the product set is and convince yourself of this statement). Let $f : \mathbb{R} \to \mathbb{R}^\infty$ be defined by $f(x) = a$, where $a : \mathbb{N} \to \mathbb{R}$ is the sequence $a_n = x$ for all $n \in \mathbb{N}$. Prove f is *not* continuous with the box topology. [Hint: Pick $\mathcal{V} = \prod_{n \in \mathbb{N}} \left(\frac{-1}{n+1}, \frac{1}{n+1}\right)$. Show there is no open set $\mathcal{U}$ containing 0 such that $f[\mathcal{U}] \subseteq \mathcal{V}$].

- (3 Points) Given the product set $\prod_{\alpha \in I} X_\alpha$, the projection function $\pi_\beta : \prod_{\alpha \in I} X_\alpha \to X_\beta$ is the function $\pi_\beta(x) = x_\beta \in X_\beta$, where x_β is the β component of x. We proved in class that projections are continuous. You may use this freely. Prove that if (Y, τ_Y) is a topological space, and if $\prod_{\alpha \in I} X_\alpha$ is given the product topology, then $g : Y \to \prod_{\alpha \in I} X_\alpha$ is continuous if and only if $\pi_\alpha \circ g$ is continuous for all $\alpha \in I$. [Hint: The set of all $\prod_{\alpha \in X} \mathcal{U}_\alpha$ where $\mathcal{U}_\alpha = X_\alpha$ for all but at most **one** $\alpha \in I$ is a subbasis for the product topology. It then suffices to show the pre-image under f of such a set is open in Y.]

- (2 Points) Using this, prove $f : \mathbb{R} \to \mathbb{R}^\infty$ defined above is continuous with the product topology.

Problem 2 Let $(X,\,\tau)$ be a topological space.

- Fréchet means for all $x, y \in X$ with $x \neq y$, there exists $\mathcal{U}, \mathcal{V} \in \tau$ with $x \in \mathcal{U}$, $x \notin \mathcal{V}$, and $y \in \mathcal{V}$, $y \notin \mathcal{U}$.

- Hausdorff means for all $x, y \in X$ with $x \neq y$ there are $\mathcal{U}, \mathcal{V} \in \tau$ such that $x \in \mathcal{U}$, $y \in \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V} = \emptyset$.

- Regular means for all $x \in X$ and all closed $\mathcal{C} \subseteq X$ with $x \notin \mathcal{C}$, there exists $\mathcal{U}, \mathcal{V} \in \tau$ such that $x \in \mathcal{U}$, $\mathcal{C} \subseteq \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V} = \emptyset$.

- Normal means for all closed $\mathcal{C}, \mathcal{D} \subseteq X$ with $\mathcal{C} \cap \mathcal{D} = \emptyset$, there exists $\mathcal{U}, \mathcal{V} \in \tau$ such that $\mathcal{C} \subseteq \mathcal{U}$, $\mathcal{D} \subseteq \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V} = \emptyset$.

We proved in class (some time ago) that a topological space $(X,\,\tau)$ is Fréchet if and only if for all $x \in X$, the set $\{\,x\,\}$ is closed. You may use this freely.

- (2 Points) Prove that if $(X,\,\tau)$ is Fréchet and regular, then it is Hausdorff.

- (2 Points) Prove that if $(X,\,\tau)$ is Fréchet and normal, then it is regular.

Authors often assume regular means Hausdorff, and normal means regular. You will now prove that unless you explicity require regular to mean regular and Hausdorff, these three notions are different.

- (2 Points) Let $X = \mathbb{R}$ and define the topology τ to be all sets $\mathcal{U} \subseteq \mathbb{R}$ such that either $\mathcal{U} = \mathbb{R}$ or $0 \notin \mathcal{U}$. This is the *excluded point topology*. Prove $(X,\,\tau)$ is normal. [Hint: What do closed sets look like in this space? what do *disjoint* closed sets $\mathcal{C}$ and $\mathcal{D}$ look like?]

- (2 Points) Prove $(X,\,\tau)$ is not regular. [Hint: Let $x = 1$ and $\mathcal{C} = \mathbb{R} \setminus \{\,1\,\}$. Is $\mathcal{C}$ closed? Are there any open sets containing $\mathcal{C}$ and not 1?]

- (2 Points) Let $X = \mathbb{Z}_2$ and $\tau = \big\{\,\emptyset,\,\{\,0\,\},\,\mathbb{Z}_2\,\big\}$. $(X,\,\tau)$ is the *Sierpinski space*. Show $(X,\,\tau)$ is not Hausdorff, not regular, but it is normal. [Hint: The set has two points, and three open sets total. You can just check all combinations.]

- (2 Points) Let $X = \mathbb{Z}_2$, and let τ be the indiscrete topology, $\tau = \{\,\emptyset,\,\mathbb{Z}_2\,\}$. Prove that $(X,\,\tau)$ is not Hausdorff, but it is regular.

The last example is a bit involved (you do not need to solve anything. Just sit back and enjoy the read). The real line has as a basis all open intervals $(a,\,b)$, $a, b \in \mathbb{R}$. The *Sorgenfrey line* has as a basis all half-open intervals $[a,\,b)$. Give $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ the product topology of the Sorgenfrey line with itself. The anti-diagonal $\Delta = \{ (x, -x) \in \mathbb{R}^2 \mid x \in \mathbb{R} \}$ and the complement $\mathbb{R}^2 \setminus \Delta$ are both closed, but any open sets $\mathcal{U}, \mathcal{V}$ with $\Delta \subseteq \mathcal{U}$ and $\mathbb{R}^2 \setminus \Delta \subseteq \mathcal{V}$ must overlap, $\mathcal{U} \cap \mathcal{V} \neq \emptyset$. This shows that the Sorgenfrey plane is *not* normal. It is, however, Hausdorff and regular.

Problem 3 Let (X, τ) be a metrizable topological space with metric d.

- (2 Points) Define the distance function as $\text{dist} : X \times \mathcal{P}(X) \to \mathbb{R}$ via:

$$\text{dist}(x, A) = \inf\big\{ d(x, y) \mid y \in A \big\} \tag{3}$$

  where inf means the *infinum*. Prove that if $\mathcal{C}$ is closed and $x \notin \mathcal{C}$, then $\text{dist}(x, \mathcal{C}) > 0$. [Hint: $\mathcal{C}$ closed means it has its limit points. If $\text{dist}(x, \mathcal{C}) = 0$, can you find a sequence in $\mathcal{C}$ that converges to x? Any contradiction?]

- (2 Points) Prove that (X, τ) is regular. [Hint: For all $y \in \mathcal{C}$, let $\mathcal{U}_y = B^{(X,\,d)}_{d(x,\,y)/2}(y)$. Can you find an open subset of $X \setminus \mathcal{C}$ that contains x and is disjoint from $\bigcup_{y \in \mathcal{C}} \mathcal{U}_y$?]

- (2 Points) Prove that (X, τ) is normal. [Hint: Given closed and disjoint $\mathcal{C}, \mathcal{D}$, use the dist function to cover $\mathcal{C}$ and $\mathcal{D}$ with open balls.]

- (2 Points) A completely normal topological space is a topological space (Y, τ_Y) such that for every subset $A \subseteq Y$ with the subspace topology τ_{Y_A}, (A, τ_{Y_A}) is normal. Prove that (X, τ) is completely normal. [Hint: What are the subspaces of a metric space?]

- (2 Points) A perfectly normal topological space is a topological space (Y, τ_Y) such that for all non-empty disjoint closed sets $\mathcal{C}, \mathcal{D} \subseteq Y$ there is a continuous function $f : Y \to [0, 1]$ (with the subspace topology from $\mathbb{R}$) such $f^{-1}[\{0\}] = \mathcal{C}$ and $f^{-1}[\{1\}] = \mathcal{D}$. Prove (X, τ) is perfectly normal. [Hint: Consider $f(x) = \text{dist}(x, \mathcal{C}) / \big(\text{dist}(x, \mathcal{C}) + \text{dist}(x, \mathcal{D})\big)$ Is this well-defined? Is it continuous? Does it do the trick?]

Problem 4 A compact topological space is a topological space (X, τ) such that for all open covers $\mathcal{O} \subseteq \tau$ of X, there is a finite subcover $\Delta \subseteq \mathcal{O}$. A locally compact topological space is a topological space (X, τ) such that for all $x \in X$ there is an open set $\mathcal{U} \in \tau$ and a subset $K \subseteq X$ such that $x \in \mathcal{U}$, $\mathcal{U} \subseteq K$, and (K, τ_K) is a compact subspace with the subspace topology.

- (2 Points) Prove that a compact Hausdorff space is regular. [Hint: From class, all closed subsets of a compact space are compact.]

- (2 Points) Prove that a compact Hausdorff space is normal. [Hint: Use the fact that a compact Hausdorff space is regular.]

- (**Bonus**: 2 Points) Prove that locally compact Hausdorff implies regular.