Point-Set Topology: Homework 4

Summer 2022

Problem 1 Locally compact has two meanings, unfortunately. And they are not equivalent. To avoid ambiguity, some authors call one notion *locally compact* and another notion *strongly locally compact.*

- Locally Compact Topological Space: A locally compact topological space is a topological space (X, τ) such that for all $x \in X$ there is an open set $\mathcal{U} \in \tau$ and a compact subset $K \subseteq X$ such that $x \in \mathcal{U}$ and $\mathcal{U} \subseteq K$.

- Strongly Locally Compact Topological Space: A strongly locally compact topological space is a topological space (X, τ) such that for all $x \in X$ there is a neighborhood basis $\mathcal{B}$ of x such that for all $\mathcal{U} \in \mathcal{B}$, $\text{Cl}_\tau(\mathcal{U})$ is compact.

Your task is to show there is no ambiguity in a Hausdorff space.

- (1 Point) Prove strongly locally compact implies locally compact (No Hausdorffness needed).

- (3 Points) Prove that if (X, τ) is Hausdorff, then it is locally compact if and only if it is strongly locally compact.

The only space I know of that is locally compact but not strongly locally compact is the one point compactification of $\mathbb{Q}$. This is compact, since it is a compactification, and hence locally compact, but not strongly locally compact. It is not Hausdorff, however.

Problem 2 You've tackled Baire's first category theorem. Every completely metrizable space (a space that comes from a complete metric) is a Baire space. That is, the intersection of countably many open and dense subsets is still dense. You will now prove Baire's second category theorem, a locally compact Hausdorff space is a Baire space.

- (2 Points) Prove that if (X, τ) is a topological space, and if for all $n \in \mathbb{N}$, $\mathcal{C}_n \subseteq X$ is a non-empty closed compact subset such that $\mathcal{C}_{n+1} \subseteq \mathcal{C}_n$, then $\bigcap_{n\in\mathbb{N}} \mathcal{C}_n$ is non-empty. [Hint: Contradiction works well here. If it is empty, can you cover $\mathcal{C}_0$ with certain sets? Does this open cover have a finite subcover?]

- (2 Points) Prove that if (X, τ) is locally compact and Hausdorff, if $x \in X$, and if $\mathcal{U} \in \tau$ is such that $x \in \mathcal{U}$, then there is a compact $K \subseteq X$ and an open $\mathcal{V} \in \tau$ such that $x \in \mathcal{V}$, $\mathcal{V} \subseteq K$, and $K \subseteq \mathcal{U}$. [Hint: Locally compact Hausdorff implies regular.]

From here, the proof is a mimicry of the idea for completely metrizable spaces. Given $\mathcal{U}_n$, $n \in \mathbb{N}$, open and dense, and $\mathcal{W} \in \tau$ non-empty, we construct nested open sets $\mathcal{V}_n \subseteq \mathcal{W} \cap \bigcap_{k=0}^{n} \mathcal{U}_k$ and compact nested non-empty sets K_n such that $K_{n+1} \subseteq \mathcal{V}_n$. Using the intersection property of the K_n, we conclude $\mathcal{W} \cap \bigcap_{n \in \mathbb{N}} \mathcal{U}_k$ is non-empty, meaning $\bigcap_{n \in \mathbb{N}} \mathcal{U}_n$ is dense. Note, the first Baire category theorem is not stronger than the second Baire category theorem, and vice-versa. The Paris plane is completely metrizable, but not locally compact. The long line is locally compact and Hausdorff, but not paracompact, and hence not metrizable, and hence not competely metrizable. Both theorems have separate applications that make them equally useful.

Problem 3 Let (X, τ) be a topological space. Prove that if $\mathcal{A} \subseteq \mathcal{P}(X)$ is locally finite (that is, every point $x \in X$ has an open set $\mathcal{U} \in \tau$ such that $x \in \mathcal{U}$ and $\mathcal{U}$ has non-empty intersection with only finitely many elements of $\mathcal{A}$), then the following are true:

- (2 Points) The set $\mathcal{A}'$ defined by:

$$\mathcal{A}' = \{\, \mathrm{Cl}_\tau(A) \mid A \in \mathcal{A} \,\} \tag{1}$$

  is locally finite as well.

- (2 Points)

$$\mathrm{Cl}_\tau\Big(\bigcup_{A \in \mathcal{A}} A\Big) = \bigcup_{A \in \mathcal{A}} \big(\mathrm{Cl}_\tau(A)\big) \tag{2}$$

Note there is no requirement that $\mathcal{A}$ covers X, nor is there a requirement that $\mathcal{A}$ consists of open sets. The only requirement is that the collection of sets is locally finite.

Problem 4 Some definitions from class.

- Basis: A basis for a topological space (X, τ) is a subset $\mathcal{B} \subseteq \tau$ such that $\mathcal{B}$ is an open cover of X, and such that $\mathcal{B}$ generates τ and for all $\mathcal{U}, \mathcal{V} \in \mathcal{B}$ and for all $x \in \mathcal{U} \cap \mathcal{V}$ there is a $\mathcal{W} \in \mathcal{B}$ such that $x \in \mathcal{W}$ and $\mathcal{W} \subseteq \mathcal{U} \cap \mathcal{V}$.

- Locally Finite Collection: A locally finite collection of sets in a topological space (X, τ) is a set $\mathcal{A} \subseteq \mathcal{P}(X)$ such that for all $x \in X$ there is a $\mathcal{U} \in \tau$ such that $x \in \mathcal{U}$ and only finitely many elements of $\mathcal{A}$ have non-empty intersection with $\mathcal{U}$.

- σ Locally Finite Collection: A σ locally finite collection of sets in a topological space (X, τ) is a set $\mathcal{A} \subseteq \mathcal{P}(X)$ such that there exists countably many sets $\mathcal{A}_n$, each of which is locally finite in (X, τ), such that $\mathcal{A} = \bigcup_{n \in \mathbb{N}} \mathcal{A}_n$.

- σ Locally Finite Basis: A σ locally finite basis of a topological space $(X,\, \tau)$ is a basis $\mathcal{B}$ of τ such that $\mathcal{B}$ is σ locally finite.
- Locally Metrizable: A locally metrizable topological space is a topological space $(X,\, \tau)$ such that for all $x \in X$ there is a $\mathcal{U} \in \tau$ such that $x \in \mathcal{U}$ and $(\mathcal{U},\, \tau_{\mathcal{U}})$ is metrizable, where $\tau_{\mathcal{U}}$ is the subspace topology.

From class, the Nagata-Smirnov theorem says $(X,\, \tau)$ is metrizable if and only if it is Hausdorff, regular, and has a σ locally finite basis. You may use this freely. **Smirnov's Theorem**: $(X,\, \tau)$ is metrizable if and only if it is Hausdorff, paracompact, and locally metrizable. One direction has already been proved. Metrizable implies Hausdorff, metrizable definitely implies locally metrizable (for each $x \in X$ pick $\mathcal{U} = X$), and metrizable implies paracompact by Stone's theorem (from class). Prove the other direction. Let $(X,\, \tau)$ be Hausdorff, locally metrizable, and paracompact. Prove it is metrizable.

- (1 Point) Why is $(X,\, \tau)$ regular?
- (1 Point) From being locally metrizable, there is an open cover $\mathcal{O} \subseteq \tau$ such that for all $\mathcal{U} \in \mathcal{O}$, $(\mathcal{U},\, \tau_{\mathcal{U}})$ is metrizable. Why is there a locally finite open refinement $\mathcal{X}$ of $\mathcal{O}$ that still covers X?
- (2 Points) Since the elements of $\mathcal{X}$ are subsets of elements of $\mathcal{O}$, the elements of $\mathcal{X}$ are also metrizable (subspaces of metrizable spaces are metrizable). So for all $\mathcal{U} \in \mathcal{X}$ there is a metric $d_{\mathcal{U}}$ that induces the subspace topology $\tau_{\mathcal{U}}$. Given $x \in \mathcal{U}$ and $\varepsilon > 0$, the open ball $B_{\varepsilon}^{(\mathcal{U},\, d_{\mathcal{U}})}(x)$ is open in $\mathcal{U}$. Why is it open in X?
- (1 Point) For all $q \in \mathbb{Q}^{+}$ let $\mathcal{A}_q$ be the set of all open balls of radius q centered about all points $x \in \mathcal{U}$ for all $\mathcal{U} \in \mathcal{X}$. This is an open cover of X for all $q \in \mathbb{Q}^{+}$. Again, for each $q \in \mathbb{Q}^{+}$ can you find a locally finite open refinement $\mathcal{Y}_q$ of $\mathcal{A}_q$ that covers X?
- (1 Points) Explain why $\mathcal{Y} = \bigcup_{q \in \mathbb{Q}^{+}} \mathcal{Y}_q$ is a σ locally finite open cover.
- (2 Points) We want to show $\mathcal{Y}$ is a basis for τ. Given $x \in X$ and $\mathcal{V} \in \tau$ with $x \in \mathcal{V}$, since $\mathcal{X}$ is locally finite, there are only finitely many sets $\mathcal{U}_0,\, \dots,\, \mathcal{U}_n$ in $\mathcal{X}$ that contain x. So $\mathcal{V} \cap \mathcal{U}_k$ is an open subset of $\mathcal{U}_k$ for all $k \in \mathbb{Z}_{n+1}$ that contains x, so there is an $\varepsilon_k > 0$ such that $B_{\varepsilon_k}^{(\mathcal{U}_k,\, d_{\mathcal{U}_k})}(x) \subseteq \mathcal{V} \cap \mathcal{U}_k$. Let $q \in \mathbb{Q}^{+}$ be less than $\min\{\, \varepsilon_k \mid k \in \mathbb{Z}_{n+1} \,\}/2$. Since $\mathcal{Y}_q$ covers X there is a set $\mathcal{W} \in \mathcal{Y}_q$ such that $x \in \mathcal{W}$. Since $\mathcal{Y}_q$ is a refinement of $\mathcal{A}_q$ there is an open ball $B_q^{(\mathcal{U},\, d_{\mathcal{U}})}(y) \in \mathcal{A}_q$ that contains $\mathcal{W}$. Show that $\mathcal{U}$ is actually one of the sets $\mathcal{U}_0,\, \dots,\, \mathcal{U}_n$. [Hint: You just need to show that $x \in \mathcal{U}$ is true].
- (2 Points) Conclude that $\mathcal{W} \subseteq \mathcal{V}$, so $\mathcal{Y}$ is a basis. Conclude that $(X,\, \tau)$ is metrizable.

Problem 5 (4 Points) The real projective space $\mathbb{RP}^n$ is the quotient of $\mathbb{R}^{n+1} \setminus \{\, \mathbf{0} \,\}$ by the equivalence relation $\mathbf{y}R\mathbf{x}$ if and only if $\mathbf{y} = \lambda\mathbf{x}$ for some $\lambda \in \mathbb{R} \setminus \{\, 0 \,\}$. Equipped with the quotient topology, $\mathbb{RP}^n$ is a topological manifold (this was proven in class). Show that $\mathbb{RP}^n$ is a compact topological manifold.

Problem 6 Prove that $(X,\, \tau)$ is a topological manifold if and only if it is locally Euclidean, Hausdorff, and σ compact.

- (2 Points) Prove a metrizable Lindelöf space is second countable. [Hint: For all $n \in \mathbb{N}$, cover the space is $1/(n+1)$ balls. Use Lindelöf to extract a countable subcover $\mathcal{B}_n$. Consider the collection of all such open sets for all $n \in \mathbb{N}$. Prove this is a countable basis.]

- (2 Points) σ compact implies Lindelöf. Using locally Euclidean, Hausdorff, and σ compact, prove $(X,\, \tau)$ is compactly exhaustible. From class, a compactly exhaustible Hausdorff space is paracompact, hence $(X,\, \tau)$ is a locally metrizable (since locally Euclidean), Hausdorff, paracompact space, so by Smirnov's theorem it is metrizable. The previous part of the problem then shows that $(X,\, \tau)$ is second countable.

Problem 7 (12 Points) This can be quite tricky, so I've made the bounty 12 points. Hopefully this pleases the masses. Let $(X,\, \tau)$ be locally Euclidean (for all $x \in X$ there is an open set $\mathcal{U} \in \tau$, $x \in \mathcal{U}$, and an injective continuous open mapping $f : \mathcal{U} \to \mathbb{R}^n$ for some $n \in \mathbb{N}$), Hausdorff, and **connected** (this last part is very important). Prove $(X,\, \tau)$ is a topological manifold if and only if it is paracompact. [Hint: All that is missing is second countability. Prove a locally Euclidean, paracompact, connected, Hausdorff space is second countable.]

Problem 8 (2 Points) Think of a space $(X,\, \tau)$ that is locally Euclidean, Hausdorff, and paracompact, but not a manifold (Note: The assumption of connectedness has been dropped). [Hint: Give $\mathbb{R}$ a topology that comes from a particular metric. Problem 7 says the space better be disconnected.]