Point-Set Topology: Homework 1

Summer 2022

Problem 1 This problem explores the use of the word *or* in mathematics. You will prove the *principle of explosion*. The principle says that if P is a statement that is both true and false, then for any statement Q, Q is true. A system that contains a sentence that is both true and false is called *inconsistent*. The principle of explosion shows that inconsistent systems are very boring.

- (2 Points) Let P be a statement that is true, and let Q be any other claim. Since P is true, what can you conclude about P *or* Q?
- (2 Points) Suppose P is also false. Using what you've concluded about P *or* Q, what can you prove about Q?

Solution. The truth table for logical or, also called *disjunction*, which is represented by $\vee$ is given in Tab. 1. We see that P *or* Q is false only when both P and Q are both false simultaneously. Since we have a statement P and we are told P is true, regardless of the statement Q we may conclude that P *or* Q is a true sentence. Using the notation of mathematical logic, we have that $P \vee Q$ is true.

Now we suppose also that P is false. Again appealing to the truth table, since $P \vee Q$ is a true statement, and since P is a false statement, Q must also be a true statement (otherwise $P \vee Q$ would be false since both P and Q are false). Therefore if P is a statement that is both true and false, and if Q is any statement, then Q is true. $\square$

P	Q	$P \vee Q$
False	False	False
False	True	True
True	False	True
True	True	True

Table 1: Truth Table for Logical Or

Problem 2 Here you will explore more set theory. You will prove the *axiom of unrestricted comprehension* is inconsistent. The axiom allows you to construct a set arbitrarily using any sentence. That is, if $P(x)$ is a sentence, you may collect all x such that $P(x)$ is true. You could write:

$$A = \{\, x \mid P(x) \,\}$$

- (1 Point) Let $P(x)$ be the sentence x *is a set*. Let A be the set $A = \{\, x \mid P(x) \,\}$. That is:
$$A = \{\, x \mid x \text{ is a set} \,\}$$
Describe the set A. Is it true that $A \in A$?

- (2 Points) Let B be the set
$$B = \{\, x \in A \mid x \notin x \,\}$$
Prove that $B \in B$. (Hint: Suppose $B \notin B$ and arrive at a contradiction).

- (2 Points) Now prove that $B \notin B$. (Hint: Same as before. Suppose $B \in B$ and arrive at a contradiction).

- (1 Point) Using the previous problem, why should we not accept the axiom of unrestricted comprehension as true?

Solution. The set A is, in plain English, the *set of all sets*. Since it is itself a set, by definition, it must be true that $A \in A$.

The set B is the set of all sets that do not contain themselves. I believe Russell called these *proper* sets, but I can't quite recall. To prove $B \in B$ we suppose not. Then $B \notin B$. But if $B \notin B$, then by the definition of B, $B \in B$, which is a contradiction. We conclude that $B \in B$.

Now we prove $B \notin B$. Again, we prove by contradiction. Suppose $B \in B$. But by definition of B, B is the set of all sets that do not contain themselves, so B can't be an element of B, a contradiction. Hence, $B \notin B$.

This shows the axiom of unrestricted comprehension is invalid. We cannot be allowed to form sets via any sentence, we must restrict ourselves to applying sentences to sets we already know exist. The principle of explosion, as demonstrated in the first problem, shows that if we allow the axiom of unrestricted comprehension, then *every* statement is both true and false. □

Problem 3 The axiom of infinity tells us $\mathbb{N} = \{0, 1, 2, \ldots\}$ exists. It does not tell us $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ exists, but we can construct it. For this problem we can assume addition (+) for natural numbers exists.

- (2 Points) Consider the set $\mathbb{N} \times \mathbb{N}$. Define R to be the relation $(a, b)R(c, d)$ if and only if $a + d = b + c$. Prove this is an equivalence relation.
- (1 Point) Describe the equivalence class of (a, b) geometrically. (Hint: $\mathbb{N} \times \mathbb{N}$ is a lattice of points in the plane. Describe the equivalence class of (a, b) using this lattice).
- (2 Points) For equivalence classes $[(a, b)]$ and $[(c, d)]$, define $[(a, b)] + [(c, d)] = [(a + c, b + d)]$. Prove this is well-defined.
- (1 Point) With this construction we now write (for convenience) $0 = [(0, 0)]$, $n = [(n, 0)]$, and $-n = [(0, n)]$. Justifiy this notation (Does $[(0, 0)]$ behave like 0? Does $[(0, n)]$ act like $-n$?)

Solution. R is reflexive. Given $(a, b) \in \mathbb{N} \times \mathbb{N}$, we have $a + b = b + a$ by the commutative law of addition, and therefore $(a, b)R(a, b)$.

R is also symmetric. If $(a, b)R(c, d)$, then $a + d = b + c$. But equality is symmetric, so $b + c = a + d$. But addition is commutative, so $c + b = d + a$. That is, $(c, d)R(a, b)$.

Lastly, R is transitive. If $(a, b)R(c, d)$, then $a + d = b + c$. If $(c, d)R(e, f)$, then $c + f = d + e$. But then:

$$\begin{aligned}
(a + f) + (c + d) &= (a + d) + (c + f) && \text{(Associativity and Commutativity)}\\
&= (b + c) + (c + f) && \text{(Substitution)}\\
&= (b + c) + (d + e) && \text{(Substitution)}\\
&= (b + e) + (c + d) && \text{(Associativity and Commutativity)}\\
\Rightarrow (a + f) + (c + d) &= (b + e) + (c + d) && \text{(Transitivity of Equality)}
\end{aligned}$$

By the cancellative property of addition, $a + f = b + e$. That is, $(a, b)R(e, f)$. Since R is reflexive, symmetric, and transtive, R is an equivalence relation.

Next, consider the equivalence class of $(0, 0)$, the set $[(0, 0)]$. This is the set of all $(m, n) \in \mathbb{N} \times \mathbb{N}$ such that $0 + n = 0 + m$, which is the same as $n = m$. This is the *diagonal* through the Cartesian plane, the set of points (n, n) in the lattice $\mathbb{N} \times \mathbb{N}$. What about $[(1, 0)]$? This is the set of points (m, n) such that $1 + n = 0 + m$, so $m = 1 + n$. This is the set of points that lie on the line of slope 1 passing through the point $(1, 0)$. In general, $[(x_0, y_0)]$ is the set of all (x, y) with $y + x_0 = x + y_0$. This property is precisely the property of a line with slope 1 passing through the point (x_0, y_0). The equivalence class of $[(m, n)]$ is a straight line through the lattice $\mathbb{N} \times \mathbb{N}$ of slope 1 containing (m, n). This is depicted in Fig. 1.

Addition of equivalence classes is well-defined here. Let $[(a,\, b)] = [(x,\, y)]$ and $[(c,\, d)] = [(z,\, w)]$. We need to show that $[(a+c,\, b+d)] = [(x+z,\, y+w)]$. Since $[(a,\, b)] = [(x,\, y)]$, and since R is an equivalence relation, we know that $(a,\, b)R(x,\, y)$. That is, $a+y = b+x$. Similarly, $(c,\, d)R(z,\, w)$, so $c+w = d+z$. Then:

$$
\begin{aligned}
a+c+y+w &= (a+y)+(c+w) && \text{(Associativity and Commutativity)}\\
&= (b+x)+(c+w) && \text{(Substitution)}\\
&= (b+x)+(d+z) && \text{(Substitution)}\\
&= b+d+x+z && \text{(Associativity and Commutativity)}\\
\Rightarrow a+c+y+w &= b+d+x+z && \text{(Transitivity of Equality)}
\end{aligned}
$$

So $(a+c,\, b+d)R(x+z,\, y+w)$, and hence $[(a+c,\, b+d)] = [(x+z,\, y+w)]$, so addition is well-defined.

Lastly, $[(0,\, 0)]$ behaves like the additive identity. This is the arithmetic property of *zero*. Given $[(m,\, n)]$, we have $[(0,\, 0)]+[(m,\, n)] = [(0+m,\, 0+n)] = [(m,\, n)]$, so addition by $[(0,\, 0)]$ does not change anything. The *number* $[(0,\, n)]$ also behaves like the negative of $[(n,\, 0)]$ since $[(n,\, 0)] + [(0,\, n)] = [(n,\, n)] = [(0,\, 0)]$. $\square$

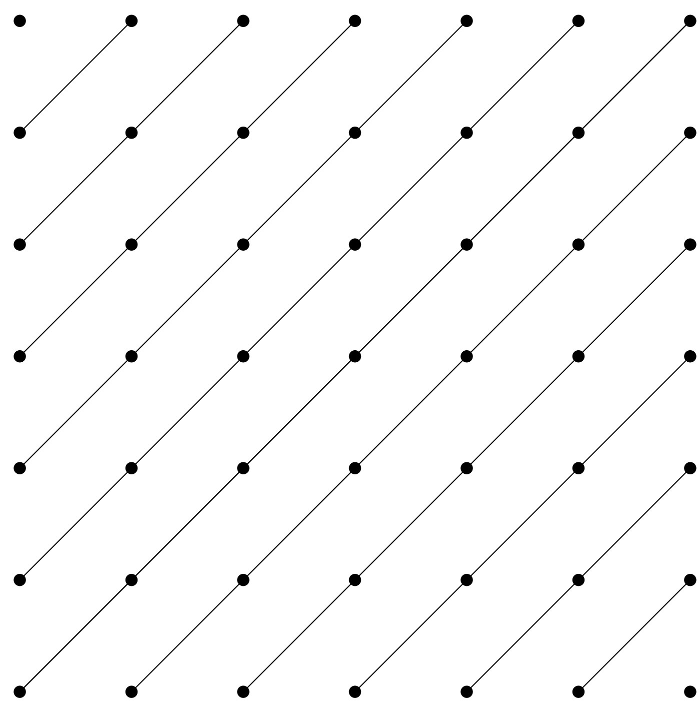

Figure 1: Equivalence Class of Points

Problem 4 Now that we have constructed $\mathbb{Z}$, let's construct $\mathbb{Q}$, the set of rational numbers. Consider the set $\mathbb{Z} \times (\mathbb{Z} \setminus \{ 0 \})$. Define the relation R by $(a, b)R(c, d)$ if and only if $ad = bc$. (We are assuming we have already constructed multiplication for integers $n \in \mathbb{Z}$).

- (2 Points) Prove R is an equivalence relation.
- (2 Points) For equivalence classes $[(a, b)]$ and $[(c, d)]$, define $[(a, b)] + [(c, d)] = [(ad + bc, bd)]$ (This is *cross-multiplying*). Prove this is well-defined.

With this we write $[(a, b)] = \frac{a}{b}$.

Solution. R is reflexive. Given (a, b) we have $ab = ba$ by the commutative property of multiplication, so $(a, b)R(a, b)$.

R is symmetric. If $(a, b)R(c, d)$, then $ad = bc$. But then $bc = ad$ since equality is symmetric. But then $cb = da$ since multiplication is commutative. Therefore, $(c, d)R(a, b)$.

Lastly, R is transitive. If $(a, b)R(c, d)$, then $ad = bc$. If $(c, d)R(e, f)$, then $cf = de$. Using this we have:

$$\begin{aligned}
(af)d &= (ad)f && \text{(Associativity and Commutativity)} \\
&= (bc)f && \text{(Substitution)} \\
&= b(cf) && \text{(Associativity)} \\
&= b(de) && \text{(Substitution)} \\
&= (be)d && \text{(Associativity and Commutativity)}
\end{aligned}$$

So $afd = bed$. But $d \in \mathbb{Z} \setminus \{ 0 \}$, so d is non-zero, and hence by the cancellation law of multiplication, $af = be$. That is, $(a, b)R(e, f)$, so R is transitive. Since R is reflexive, symmetric, and transitive, it is an equivalence relation.

Addition of equivalence classes is well-defined here. Given $[(a, b)] = [(x, y)]$ and $[(c, d)] = [(z, w)]$, we have, since R is an equivalence relation, that $(a, b)R(x, y)$ and $(c, d)R(z, w)$. That is, $ay = bx$ and $cw = dz$. We need to show that $[(ad + bc, bd)] = [(xw + yz, yw)]$, meaning we need to show that $(ad + bc)yw =$

$(xw + yz)bd$. We have:

$$
\begin{aligned}
(ad + bc)yw &= (ad)(yw) + (bc)(yw) && \text{(Distributivity)} \\
&= (ay)(dw) + (bc)(yw) && \text{(Associativity and Commutativity)} \\
&= (bx)(dw) + (bc)(yw) && \text{(Substitution)} \\
&= (bx)(dw) + (cw)(by) && \text{(Associativity and Commutativity)} \\
&= (bx)(dw) + (dz)(by) && \text{(Substitution)} \\
&= (xw)(bd) + (dz)(by) && \text{(Associativity and Commutativity)} \\
&= (xw)(bd) + (yz)(bd) && \text{(Associativity and Commutativity)} \\
&= (xw + yz)bd && \text{(Distributivity)}
\end{aligned}
$$

So $[(ad + bc,\, bd)] = [(xw + yz,\, yw)]$, and hence addition is well-defined. $\square$

Problem 5 Prove Cantor's theorem. If A is a set, and $\mathcal{P}(A)$ is the power set of A, then there is no surjection $f : A \to \mathcal{P}(A)$.

- (1 Point) Suppose there is a surjection $f : A \to \mathcal{P}(A)$. Consider the set $B = \{ x \in A \mid x \notin f(x) \}$. Describe in words what the set B contains.
- (2 Points) Since $B \subseteq A$, and since f is surjective, there is an element $a \in A$ such that $f(a) = B$. Show that this is a contradiction.
- (1 Point) Construct an injective function $g : A \to \mathcal{P}(A)$. (Hint: Given $a \in A$, what's an "obvious" subset we can send a to?)

Solution. The set B is the set of all elements $a \in A$ such that the image of a under f does not contain a. That is, since f takes elements of A and returns subsets of A, it is possible for a to be an element of $f(a)$. Consider $\mathbb{Z}_3$ and define $f : \mathbb{Z}_3 \to \mathcal{P}(\mathbb{Z}_3)$ by $f(0) = \{ 1, 2 \}$, $f(1) = \{ 0, 1 \}$, and $f(2) = \emptyset$. Then $0 \notin f(0)$ since $f(0)$ only contains 1 and 2. We see that $1 \in f(1)$ since $f(1)$ is a set containing 1. Lastly, $2 \notin f(2)$ since $f(2)$ is the empty set. The set B consisting of all elements $a \in \mathbb{Z}_3$ such that $a \notin f(a)$ is $\{ 0, 2 \}$. Note that there is no element $n \in \mathbb{Z}_3$ such that $f(n) = B$.

Suppose $f : A \to \mathcal{P}(A)$ is a surjection and B is defined as above. Since f is surjective, there is an $a \in A$ such that $f(a) = B$. Suppose $a \in B$. Then $a \in f(a)$ since $f(a) = B$, a contradiction since B is the set of all $x \in A$ such that $x \notin f(x)$. So $a \notin B$. But if $a \notin B$, then $a \notin f(a)$ since $f(a) = B$. But if $a \notin f(a)$, then $a \in B$ since B is the set of all $x \in A$ such that $x \notin f(x)$. This is a contradiction meaning a does not exist, so f is not a surjection.

There is an injective function $f : A \to \mathcal{P}(A)$. Define $f(a) = \{ a \}$ for all $a \in A$. If $f(a) = f(b)$, then $\{ a \} = \{ b \}$, meaning $a = b$, so f is injective. $\square$

Problem 6 We proved in class that a function $f : X \to Y$ from a metric space (X, d_X) to a metric space (Y, d_Y) is continuous if and only if for every open subset $\mathcal{V} \subseteq Y$, the pre-image $f^{-1}[\mathcal{V}]$ is also open. You will now prove the equivalence of the third definition of continuity.

- (3 Points) Prove that if f is continuous, then for all $\varepsilon > 0$, and for all $x \in X$, there is a $\delta > 0$ such that if $x_0 \in X$ and $d_X(x, x_0) < \delta$, then $d_Y\big(f(x), f(x_0)\big) < \varepsilon$ (Hint: Suppose not. Then there is an $\varepsilon > 0$ such that for each $n \in \mathbb{N}$, $n > 0$, there is a point $a_n \in X$ with $d_X(x, a_n) < \frac{1}{n}$ and $d_Y\big(f(x), f(a_n)\big) \geq \varepsilon$. What is $\lim_{n\to\infty} a_n$? What is $\lim_{n\to\infty} f(a_n)$? Is there a contradiction?)

- (3 Points) Prove that if $f : X \to Y$ is a function such that for all $\varepsilon > 0$ and for all $x \in X$, there is a $\delta > 0$ such that for all $x_0 \in X$, $d_X(x, x_0) < \delta$ implies $d_Y\big(f(x), f(x_0)\big) < \varepsilon$, then f is continuous. (Hint: Let $a_n \to x$ by a convergent sequence. What does this property say about $\lim_{n\to\infty} f(a_n)$?)

Solution. Suppose $f : X \to Y$ is continuous. Suppose there is an $\varepsilon > 0$ and an $x \in X$ such that for all $\delta > 0$ there is an $x_0 \in X$ such that $d_X(x,\, x_0) < \delta$ and $d_Y\big(f(x),\, f(x_0)\big) \geq \varepsilon$. Then, in particular, for all $n \in \mathbb{N}$ there is an $a_n \in X$ such that $d_X(x,\, a_n) < \frac{1}{n+1}$, but $d_Y\big(f(x),\, f(a_n)\big) \geq \varepsilon$. But then $d_X(x,\, a_n) \to 0$, and so $a_n \to x$. But f is continuous, so if $a_n \to x$, then $f(a_n) \to f(x)$. That is, by the definition of convergence, there is an $N \in \mathbb{N}$ such that $n \in \mathbb{N}$ and $n > N$ implies $d_Y\big(f(x),\, f(a_n)\big) < \varepsilon$. But for all $n \in \mathbb{N}$ we have $d_Y\big(f(x),\, f(a_n)\big) \geq \varepsilon$ which is a contradiction. Hence, for all $\varepsilon > 0$ and for all $x \in X$ there is a $\delta > 0$ such that for all $x_0 \in X$ it is true that $d_X(x,\, x_0) < \delta$ implies $d_Y\big(f(x),\, f(x_0)\big) < \varepsilon$.

Now suppose f has the property that for all $x \in X$ and for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x_0 \in X$, $d_X(x,\, x_0) < \delta$ implies $d_Y\big(f(x),\, f(x_0)\big) < \varepsilon$. Suppose f is not continuous. Then there is a convergent sequence $a : \mathbb{N} \to X$ and an $x \in X$ such that $a_n \to x$, but $f(a_n) \not\to f(x)$. But then there is an $\varepsilon > 0$ such that for all $N \in \mathbb{N}$ there is an $n \in \mathbb{N}$ with $n > N$ such that $d_Y\big(f(x),\, f(a_n)\big) \geq \varepsilon$. But by the property of f, since $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x_0 \in X$, $d_X(x,\, x_0) < \delta$ implies $d_Y\big(f(x),\, f(x_0)\big)$. But $a_n \to x$ and $\delta > 0$, so there is an $N \in \mathbb{N}$ such that $n \in \mathbb{N}$ and $n > N$ implies $d_X(x,\, a_n) < \delta$. But then for all $n > N$ we have $d_Y\big(f(x),\, f(a_n)\big) < \varepsilon$, a contradiction. Hence, f is continuous. $\square$

Problem 7 A locally compact metric space is a metric space $(X,\, d)$ where for all $x \in X$ there is a compact set K and an open set $\mathcal{U}$ such that $x \in \mathcal{U}$ and $\mathcal{U} \subseteq K$ (See Fig. 2).

- (2 Points) Construct a metric space that is *not* locally compact. Explain why it is not locally compact. (Hint: Il est utile de penser á la France).

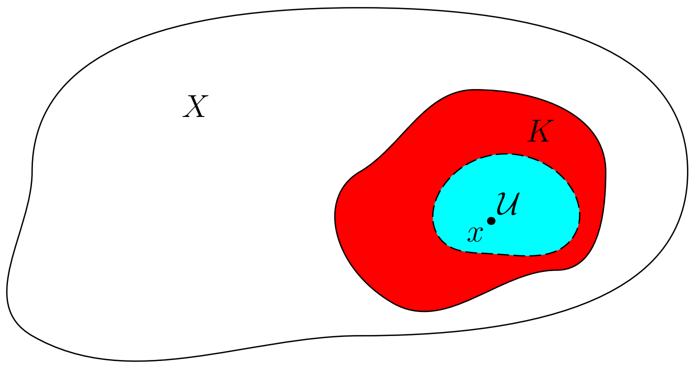

Figure 2: Diagram for Locally Compact Metric Spaces

Let $X = \mathbb{R}^2$ and $d_P : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be the Paris metric:

$$d_P(\mathbf{x},\, \mathbf{y}) = \begin{cases} ||\mathbf{x} - \mathbf{x}||_2 & \mathbf{y} = \lambda\mathbf{x} \text{ for some } \lambda \in \mathbb{R} \\ ||\mathbf{x}||_2 + ||\mathbf{y}||_2 & \text{otherwise} \end{cases} \tag{1}$$

For every point $\mathbf{x} \neq \mathbf{0}$, the point $\mathbf{x}$ has the local compactness property. Choosing $r = ||\mathbf{x}||_2/2$, the open ball of radius r centered at $\mathbf{x}$ in the Paris plane is essentially an *open interval*, topologically similar to the open unit interval $(0,\, 1)$. By adding the end points of this interval, that is, the points two points $\mathbf{y}$ lying on the line between the origin and $\mathbf{x}$ such that $||\mathbf{x} - \mathbf{y}||_2 = r$, then we get a set that topologically looks like the closed interval $[0,\, 1]$, which is compact by the Heine-Borel theorem. So we have found an open set $\mathcal{U}$ and a compact set K such that $\mathbf{x} \in \mathcal{U}$ and $\mathcal{U} \subseteq K$. If we are going to prove the Paris plane is not locally compact, we're going to have to look at the origin. Suppose the Paris plane is locally compact. Then there is an open set $\mathcal{U} \subseteq \mathbb{R}^2$ and a compact set $K \subseteq \mathbb{R}^2$ such that $\mathbf{0} \in \mathcal{U}$ and $\mathcal{U} \subseteq K$. But since $\mathbf{0} \in \mathcal{U}$ and $\mathcal{U}$ is open, there is an $\varepsilon > 0$ such that $B_\varepsilon^{(\mathbb{R}^2,\, d_P)}(\mathbf{0}) \subseteq \mathcal{U}$. Since $\mathcal{U} \subseteq K$, we have $B_\varepsilon^{(\mathbb{R}^2,\, d_P)}(\mathbf{0}) \subseteq K$. Define $a : \mathbb{N} \to K$ by:

$$a_n = \frac{\varepsilon}{2}\Big(\cos\big(\frac{\pi}{n+1}\big),\, \sin\big(\frac{\pi}{n+1}\big)\Big) \tag{2}$$

This function is injective since $f(n) = \frac{\pi}{n+1}$ is injective, and $\big(\cos(t),\, \sin(t)\big)$ is injective for $t \in [0,\, \pi]$. Moreover, since every $n \in \mathbb{N}$ corresponds to a different

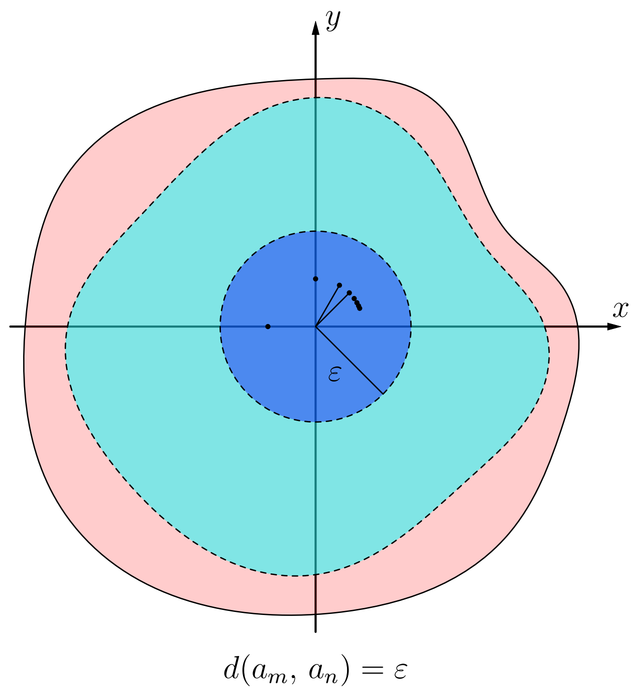

Figure 3: Non-Convergent Sequence in the Paris Plane

angle, no two points a_n and a_m lie on the same line through the origin. But then, for all $n, m \in \mathbb{N}$, we have:

$$d_P(a_m,\, a_n) = ||a_m||_2 + ||a_n||_2 = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \tag{3}$$

That is, $a : \mathbb{N} \to \mathbb{R}^2$ is not a Cauchy sequence since the distance betwen distinct points does not converge to zero, but rather is a constant. Moreover, no subsequence can converge since for any subsequence a_k, since k must be strictly increasing, we also have $d_P(a_{k_m},\, a_{k_n}) = \varepsilon$ for all $m \neq n$. We have constructed a sequence that has no convergent subsequence. But $a : \mathbb{N} \to K$ is a sequence in a compact set, so it must have a convergent subsequence, which is a contradiction. Therefore, $(\mathbb{R}^2,\, d_P)$ is *not* locally compact. See Fig. 3.