Point-Set Topology: Homework 2

Summer 2022

Problem 1 A few more notes about metric spaces. A *contraction* on a metric space $(X,\, d)$ is a function $f : X \to X$ such that for all $x, y \in X$ it is true that $d\big(f(x),\, f(y)\big) \leq r\, d(x,\, y)$ for some fixed $0 \leq r < 1$. This means the function f *squeezes* the points together. You will prove one of the most celebrated theorems of the theory of metric spaces, the *Banach Fixed Point Theorem*. If $(X,\, d)$ is a non-empty complete metric space, and if $f : X \to X$ is a contraction, then there is a unique point $x \in X$ such that $f(x) = x$. That is, f has a unique *fixed-point*, a point that is not changed by f.

- (2 Points) Prove that a contraction $f : X \to X$ is continuous.
- (2 Points) Prove that if $f : X \to X$ has a fixed-point $x \in X$, then x is the only fixed-point. [Hint: What if $y \in X$ is another fixed-point? Anything wrong?]
- (2 Points) Let $a_0 \in X$ be arbitrary, define a_n inductively via $a_{n+1} = f(a_n)$. Prove that for all $n \in \mathbb{N}$, $d(a_{n+1},\, a_n) \leq r^n d(a_1,\, a_0)$, where $0 \leq r < 1$ is a value such that for all $x, y \in X$ we have $d\big(f(x),\, f(y)\big) \leq r\, d(x,\, y)$.
- (2 Points) Conclude that $a : \mathbb{N} \to X$ is a Cauchy sequence. [Hint: Apply the triangle inequality and use the geometric series from calculus].
- (2 Points) Since $(X,\, d)$ is complete, the sequence converges. Let $x \in X$ be such that $a_n \to x$. Show that $f(x) = x$. [Hint: Use the continuity of f that you proved in the first part of this problem]

Solution. A contraction is continuous. Let $a : \mathbb{N} \to X$ be a convergent sequence and let $x \in X$ be a limit of a. That is:

$$\lim_{n\to\infty} d(x,\, a_n) = 0 \tag{1}$$

But since f is a contraction there is an $r \in [0,\, 1)$ such that for all $a, b \in X$ we have $d\big(f(a),\, f(b)\big) \leq r\, d(a,\, b)$. But then:

$$\lim_{n\to\infty} d\big(f(x),\, f(a_n)\big) \leq \lim_{n\to\infty} r\, d(x,\, a_n) = 0 \tag{2}$$

so $f(a_n) \to f(x)$, and therefore f is continuous.

If $f : X \to Y$ is a contraction, and if $x \in X$ is a fixed-point, then it is the only fixed-point. Suppose $y \in X$ is a different fixed-point, $x \neq y$. Then:

$$d(x,\, y) = d\big(f(x),\, f(y)\big) \le r\, d(x,\, y) < d(x,\, y) \tag{3}$$

so $d(x,\, y) < d(x,\, y)$, which is a contradiction. So x is the unique fixed-point if it exists.

We can prove $d(a_{n+1},\, a_n) \le r^n d(a_1,\, a_0)$ by induction. The base case is true since f is a contraction. That is:

$$d(a_2,\, a_1) = d\big(f(a_1),\, f(a_0)\big) \le r\, d(a_1,\, a_0) \tag{4}$$

Now suppose the claim is true for $n \in \mathbb{N}$. We must prove this implies the claim is true for $n+1$. We have:

$$d(a_{n+2},\, a_{n+1}) = d\big(f(a_{n+1}),\, f(a_n)\big) \le r\, d(a_{n+1},\, a_n) \le r^{n+1} d(a_1,\, a_0) \tag{5}$$

where this last inequality follows from the induction hypothesis. Therefore, by the principle of induction, the claim is true for all $n \in \mathbb{N}$.

This inequality can be used to prove that $a : \mathbb{N} \to X$ is a Cauchy sequence. Repeatedly using the triangle inequality, if $m, n \in \mathbb{N}$ and $m < n$, we have:

$$\begin{aligned} d(a_m,\, a_n) &\le d(a_m,\, a_{m+1}) + d(a_{m+1},\, a_n) && (6)\\ &\le d(a_m,\, a_{m+1}) + d(a_{m+1},\, a_{m+2}) + d(a_{m+2},\, a_n) && (7) \end{aligned}$$

Inductively we obtain:

$$d(a_m,\, a_n) \le \sum_{k=m}^{n-1} d(a_k,\, a_{k+1}) \tag{8}$$

Invoking the inequality we just proved, we get:

$$d(a_m,\, a_n) \le \sum_{k=m}^{n-1} d(a_k,\, a_{k+1}) \le \sum_{k=m}^{n-1} r^k\, d(a_0,\, a_1) \tag{9}$$

We can simplify this and use the geometric series.

$$\begin{aligned} d(a_m,\, a_n) &\le \sum_{k=m}^{n-1} r^k\, d(a_0,\, a_0) && (10)\\ &= d(a_0,\, a_1) \sum_{k=m}^{n-1} r^k && (11)\\ &\le d(a_0,\, a_1) \sum_{k=m}^{\infty} r^k && (12)\\ &= d(a_0,\, a_1) \frac{r^m}{1-r} && (13) \end{aligned}$$

But $0 \leq r < 1$, so r^m converges to zero. Given $\varepsilon > 0$, choose N such that $d(a_0, a_1)\frac{r^N}{1-r} < \varepsilon/2$. Then, choosing $m, n > N$, we get $d(a_m, a_n) < \varepsilon$ showing us that a is a Cauchy sequence.

Since (X, d) is complete, there is some $x \in X$ such that $a_n \to x$. But then, since f is continuous, we have:

$$\begin{aligned} x &= \lim_{n\to\infty} a_{n+1} && (14)\\ &= \lim_{n\to\infty} f(a_n) && (15)\\ &= f\big(\lim_{n\to\infty} a_n\big) && (16)\\ &= f(x) && (17) \end{aligned}$$

so x is a fixed-point. □

Problem 2 A dense subset of a topological space $(X,\,\tau)$ is a subset $A\subseteq X$ such that $\text{Cl}_\tau(A)=X$. That is, every point in X is a limit point of A. For example, the rationals $\mathbb{Q}$ are a dense subset of the reals $\mathbb{R}$. A Baire topological space is a topological space $(X,\,\tau)$ such that for any non-empty countable set $\mathcal{O}\subseteq\tau$ with the property that $\mathcal{U}\in\mathcal{O}$ implies $\mathcal{U}$ is dense, the intersection $\bigcap\mathcal{O}$ is also dense. Here you will prove the first of Baire's Category Theorems (Note: The Baire category theorem has absolutely nothing to do with category theory. The terminology for this theorem came long before category theory was initiated). If $(X,\,d)$ is a complete metric space, and if τ_d is the metric topology, then $(X,\,\tau_d)$ is a Baire topological space.

- (2 Points) Prove that, for a topological space $(Y,\,\tau_Y)$, $A\subseteq Y$ is dense if and only if for every non-empty open set $\mathcal{U}\subseteq Y$, the intersection $\mathcal{U}\cap A$ is non-empty.

- (2 Points) It now suffices to prove that if $\mathcal{W}\subseteq X$ is open and non-empty, then $\mathcal{W}\cap\bigcap\mathcal{O}$ is non-empty. Show that if $\mathcal{V}$ is an open ball, $\mathcal{V}=B_r^{(X,\,d)}(x)$, then there is an $\varepsilon>0$ such that $\text{Cl}_\tau\big(B_\varepsilon^{(X,\,d)}(x)\big)\subseteq\mathcal{V}$. That is, there is always a *closed ball* inside of an open ball.

- (2 Points) Since $\mathcal{O}$ is countable, there is a surjective sequence $\mathcal{U}:\mathbb{N}\to\mathcal{O}$. That is, we may list the elements of $\mathcal{O}$ as $\mathcal{U}_0$, $\mathcal{U}_1$, and so on. Since $\mathcal{U}_0$ is open and dense, $\mathcal{U}_0\cap\mathcal{W}$ is non-empty. Hence there an $a_0\in\mathcal{U}_0\cap\mathcal{W}$. Since the intersection of open sets is open, there is a positive $r_0<1$ such that $B_{r_0}^{(X,\,d)}(a_0)\subseteq\mathcal{U}_0\cap\mathcal{W}$. By the previous part of the problem, there is a positive $\varepsilon_0<r_0$ such that $\text{Cl}_{\tau_d}\big(B_{\varepsilon_0}^{(X,\,d)}(a_0)\big)\subseteq B_{r_0}^{(X,\,d)}(a_0)$. Recursively we may define a_n, r_n, and ε_n such that $r_n<\frac{1}{n+1}$, and:

$$\text{Cl}_{\tau_d}\big(B_{\varepsilon_n}^{(X,\,d)}(a_n)\big)\subseteq B_{r_n}^{(X,\,d)}(a_n)\subseteq\mathcal{W}\cap\bigcap_{k=0}^{n}\mathcal{U}_n \tag{18}$$

  and such that:

$$\text{Cl}_{\tau_d}\big(B_{\varepsilon_{n+1}}^{(X,\,d)}(a_{n+1})\big)\subseteq B_{\varepsilon_n}^{(X,\,d)}(a_n) \tag{19}$$

  Show that $a:\mathbb{N}\to X$ is a Cauchy sequence.

- (2 Points) Since $(X,\,d)$ is complete, there is an $x\in X$ such that $a_n\to x$. Show that for all $n\in\mathbb{N}$ it is true that $x\in\mathcal{U}_n$. [Hint: Since $\text{Cl}_{\tau_d}\big(B_{\varepsilon_n}^{(X,\,d)}(a_n)\big)$ is closed, it contains all of its limit points. Show that x is a limit point of this for all n. Conclude that x is in $\mathcal{U}_n$ since $\text{Cl}_{\tau_d}\big(B_{\varepsilon_n}^{(X,\,d)}(a_n)\big)\subseteq\mathcal{U}_n$.

- (2 Points) Show that $x\in\mathcal{W}$ as well, and therefore $x\in\mathcal{W}\cap\bigcap\mathcal{O}$, proving the intersection is non-empty, and therefore $\bigcap\mathcal{O}$ is dense.

Solution. Suppose $A\subseteq Y$ is dense. Let $\mathcal{U}\in\tau_Y$ be non-empty and suppose $A\cap\mathcal{U}=\emptyset$. Then $Y\setminus\mathcal{U}$ is a closed set that contains A. But then, since

$A \subseteq Y \setminus \mathcal{U}$, and since $Y \setminus \mathcal{U}$ is closed, we have $\mathrm{Cl}_\tau(A) \subseteq Y \setminus \mathcal{U}$. But this is a contradiction since A is dense, meaning $\mathrm{Cl}_\tau(A) = Y$, but $\mathcal{U}$ is non-empty, so $Y \setminus \mathcal{U} \neq Y$. Hence, $A \cap \mathcal{U}$ is non-empty.

Now, suppose for every non-empty $\mathcal{U} \in \tau_Y$ we have that $A \cap \mathcal{U} \neq \emptyset$. Suppose $y \in Y$ is such that $y \notin \mathrm{Cl}_\tau(A)$. Then, by the definition of closure, there is a closed set $\mathcal{C} \subseteq Y$ such that $\mathrm{Cl}_\tau(A) \subseteq \mathcal{C}$ and $y \notin \mathcal{C}$. But if $\mathcal{C}$ is closed, then $Y \setminus \mathcal{C}$ is open. But since $\mathrm{Cl}_\tau(A) \subseteq \mathcal{C}$ and $A \subseteq \mathrm{Cl}_\tau(A)$, we have that $A \cap (Y \setminus \mathcal{C}) = \emptyset$. But $Y \setminus \mathcal{C}$ is non-empty since $y \in Y \setminus \mathcal{C}$. But all non-empty open subsets of Y have non-empty intersection with A, which is a contradiction. So A is dense.

The closure of an open ball is contained in a closed ball. The closed ball of radius ε in $(X,\, d)$ centered at $x \in X$ is defined by:

$$\overline{B}_\varepsilon^{(X,\,d)}(x) = \{\, y \in X \mid d(x,\, y) \leq \varepsilon \,\} \tag{20}$$

Slight change from the open ball, we've replaced $<$ with $\leq$ in the definition. Firstly, closed balls are closed. Given $\varepsilon > 0$, $x \in X$, and $y \notin \overline{B}_\varepsilon^{(X,\,d)}(x)$, choose $r = d(x,\, y) - \varepsilon$. Since y is not in the closed ball centered at x of radius ε we see that $d(x,\, y) > \varepsilon$, so $d(x,\, y) - \varepsilon$ is positive. Suppose $z \in B_r^{(X,\,d)}(y) \cap \overline{B}_\varepsilon^{(X,\,d)}(x)$. Then:

$$d(x,\, y) \leq d(x,\, z) + d(z,\, y) < \varepsilon + d(x,\, y) - \varepsilon = d(x,\, y) \tag{21}$$

so $d(x,\, y) < d(x,\, y)$, which is a contradiction. Hence $B_r^{(X,\,d)}(y) \cap \overline{B}_\varepsilon^{(X,\,d)}(x) = \emptyset$. But then the complement of $\overline{B}_\varepsilon^{(X,\,d)}(x)$ is open, meaning $\overline{B}_\varepsilon^{(X,\,d)}(x)$ is closed. Given $\varepsilon > 0$, we then have:

$$B_\varepsilon^{(X,\,d)}(x) \subseteq \mathrm{Cl}_\tau\big(B_\varepsilon^{(X,\,d)}(x)\big) \subseteq \overline{B}_\varepsilon^{(X,\,d)}(x) \tag{22}$$

Choosing $\varepsilon = r/2$ we have:

$$\mathrm{Cl}_\tau\big(B_\varepsilon^{(X,\,d)}(x)\big) \subseteq \overline{B}_\varepsilon^{(X,\,d)}(x) \subseteq B_r^{(X,\,d)}(x) \tag{23}$$

NOTE: This does not reverse, in general. The closure of the open ball does not need to be exactly the closed ball, just a subset of it. Take X to be any set and d the discrete metric. Given $x \in X$, the open ball of radius 1 centered at x is just $\{\, x \,\}$. There are no other points y with $d(x,\, y) < 1$. The closure of this is also $\{\, x \,\}$. However, the closed ball of radius 1 is all of X. Every point $y \in X$ is such that $d(x,\, y) \leq 1$.

The sequence $a : \mathbb{N} \to X$ constructed is Cauchy. Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $N + 1 > 1/\varepsilon$. Then, for $m, n > N$ with $m < n$ we have, since the open balls are nested, the following:

$$d(a_m,\, a_n) < \frac{1}{m+1} < \frac{1}{N+1} < \varepsilon \tag{24}$$

so the sequence is Cauchy.

Given $N \in \mathbb{N}$, for $n > N$ the points a_n lie entirely in $\mathrm{Cl}_\tau\big(B_{\varepsilon_n}^{(X,\,d)}(a_n)\big)$, which is closed. Since it is a Cauchy sequence and $(X,\, d)$ is complete, the sequence converges. But closed sets contain their limit points, so the limit x is contained in $\mathrm{Cl}_\tau\big(B_{\varepsilon_n}^{(X,\,d)}(a_n)\big)$. But $\mathrm{Cl}_\tau\big(B_{\varepsilon_n}^{(X,\,d)}(a_n)\big) \subseteq \mathcal{U}_n$, so the limit is contained in $\mathcal{U}_n$ as well. Since this is true of all $n \in \mathbb{N}$, we have that $x \in \bigcap_{n=0}^{\infty} \mathcal{U}_n = \bigcap \mathcal{O}$.

The closures of these open balls are also constructed so that they are contained inside of $\mathcal{W}$ for each $n \in \mathbb{N}$, see the recursive definition above. Meaning the limit is also contained in $\mathcal{W}$, and hence $\mathcal{W} \cap \bigcap \mathcal{O}$ is non-empty. □

Problem 3 From class, a Kolmogorov topology on a set X is a topology τ on X such that for all $x, y \in X$, there is an open set $\mathcal{U} \in \tau$ such that either $x \in \mathcal{U}$ and $y \notin \mathcal{U}$, or $x \notin \mathcal{U}$ and $y \in \mathcal{U}$. That is, a Kolmogorov topology is a topology where it is always possible to tell two points apart using open sets.

- (2 Points) There are 8,977,053,873,043 distinct topologies on the set $\mathbb{Z}_{10}$, 6,611,065,248,783 Kolmogorov topologies, and 4,717,687 topologies that are not homeomorphic. Quite a lot. It would be cruel to ask you to find them all. Instead, find all distinct topologies on $\mathbb{Z}_2$ (there are 4), all distinct Kolmogorov topologies (there's 3), all non-homeomorphic topologies (3), all non-homeomorphic Kolmogorov topologies (2), and all Hausdorff topologies (1). [Hint: This may seem like a lot, but it really isn't. Find the 4 topologies on $\mathbb{Z}_2$. Then examine which are Kolmogorov and which are homeomorphic, etc.]
- (2 Points) On $\mathbb{Z}_3$ there are 29 distinct topologies, 19 distinct Kolmogorov topologies, 9 non-homeomorphic topologies, and 5 non-homeomorphic Kolmogorov topologies. Find 2 non-homeomorphic Kolmogorov topologies. [Hint: Hausdorff implies Kolmogorov. Can you find the Hausdorff topology?]

Solution. The four topologies on $\mathbb{Z}_2$ are given pictorial in Fig. 1. They are the indiscrete topology $\tau_0 = \{\, \emptyset, \mathbb{Z}_2 \,\}$, the topology $\tau_1 = \{\, \emptyset, \{\, 0 \,\}, \mathbb{Z}_2 \,\}$, the topology $\tau_2 = \{\, \emptyset, \{\, 1 \,\}, \mathbb{Z}_2 \,\}$, and the discrete topology $\tau_3 = \{\, \emptyset, \{\, 0 \,\}, \{\, 1 \,\}, \mathbb{Z}_2 \,\}$. All topologies but the indiscrete topology are Kolmogorov since all others can topologically distinguish 0 and 1 via open sets. The indiscrete topology is not Kolmogorov, that points 0 and 1 are topologically indistringuishable in this topology. The topologies τ_1 and τ_2 are essentially the same, we've just relabelled 0 and 1, and indeed these topologies are homeomorphic on $\mathbb{Z}_2$. The discrete topology is the only Hausdorff topology on $\mathbb{Z}_2$.

For $\mathbb{Z}_3$, we can use the fact that in a Hausdorff topological space singleton sets $\{\, x \,\}$ are closed. Since the finite union of closed sets is closed, the only Hausdorff topology on a finite set is the discrete topology. So, $\mathcal{P}(\mathbb{Z}_3)$ is a Hausdorff, and hence Kolmogorov, topology on $\mathbb{Z}_3$. We can find another by modifying an idea from class. The topology generated on $\mathbb{N}$ by all sets of the form $\mathbb{Z}_n$ with $n \in \mathbb{N}$ can be modified to give a topology on $\mathbb{Z}_3$. Declare $\tau = \{\, \mathbb{Z}_0, \mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_3 \,\}$. This is a topology since the sets are all nested, so the intersection and union properties are satisfied, but also $\emptyset = \mathbb{Z}_0 \in \tau$ and $\mathbb{Z}_3 \in \tau$. It is Kolmogorov as well. Given $m, n \in \mathbb{Z}_3$ with $m < n$, $m \in \mathbb{Z}_n$ but $n \notin \mathbb{Z}_n$. $\square$

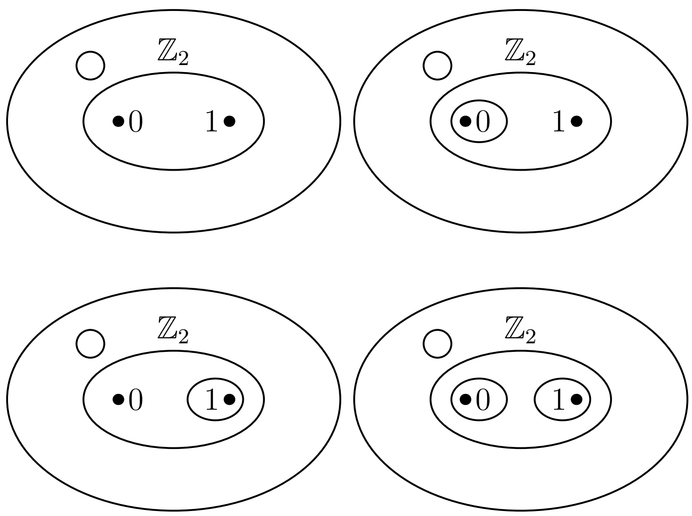

Figure 1: Topologies on $\mathbb{Z}_2$

Problem 4 (4 Points) Let (X, τ) be a sequential topological space and R an equivalence relation on X. Prove that the quotient space $(X/R, \tau_{X/R})$ is sequential as well.

Solution. Suppose not and let $\tilde{\mathcal{U}} \subseteq X/R$ be sequentially open but not open. Let $q : X \to X/R$ be the quotient map, $q(x) = [x]$. Then, by the definition of the quotient topology, $q^{-1}[\tilde{\mathcal{U}}]$ is not open since $\tilde{\mathcal{U}}$ is not open. Let $\mathcal{U} = q^{-1}[\tilde{\mathcal{U}}]$. But (X, τ) is sequential, so if $\mathcal{U}$ is not open, then it is not sequentially open. But then there is a convergent sequence $a : \mathbb{N} \to X$ that converges to a point $x \in \mathcal{U}$ such that for all $N \in \mathbb{N}$ there is an $n \in \mathbb{N}$ with $n > N$ such that $a_n \notin \mathcal{U}$. But the quotient map q is continuous, and continuous functions are sequentially continuous, so if $a_n \to x$, then $q(a_n) \to q(x)$. But then $q(a_n)$ is a convergent sequence in X/R that converges to a point $q(x) \in \tilde{\mathcal{U}}$. But $\tilde{\mathcal{U}}$ is sequentially open, so there is an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n > N$ it is true that $q(a_n) \in \tilde{\mathcal{U}}$. But then, by the definition of pre-image, we have $a_n \in \mathcal{U}$ for all $n > N$, a contradiction. Hence, $\tilde{\mathcal{U}}$ is open and $(X/R, \tau_{X/R})$ is sequential. $\square$

Problem 5 Kazimeirz Kuratowski gave an alternative, but equivalent, definition of topology. To him the notion of *closure* was sufficient to describe topological spaces. A Kuratowski closure operator on a set X is a function $\sigma : \mathcal{P}(X) \to \mathcal{P}(X)$ such that, for all $A, B \subseteq X$:

1. $\sigma(\emptyset) = \emptyset$
2. $A \subseteq \sigma(A)$
3. $\sigma(A) = \sigma\big(\sigma(A)\big)$
4. $\sigma(A \cup B) = \sigma(A) \cup \sigma(B)$

A Kuratowski space is an ordered pair $(X,\, \sigma)$ where X is a set and σ is a Kuratowski closure operator on X. We have seen in class that, if $(X,\, \tau)$ is a topological space, then Cl_τ is a Kuratowski closure operator. Now, let's go the other way.

- (2 Points) Show that, given $(X,\, \sigma)$, the set τ_σ defined by:

$$\tau_\sigma = \{\, X \setminus \mathcal{C} \in \mathcal{P}(X) \mid \sigma(\mathcal{C}) = \mathcal{C} \,\} \tag{25}$$

  is a topology on X. (We proved that, in topological spaces, $A \subseteq X$ being closed is equivalent to $\text{Cl}_\tau(A) = A$. We are intuitively defining τ_σ as the set of all *complements of closed sets*).

- (6 Points) If $(X,\, \sigma_X)$ and $(Y,\, \sigma_Y)$ are Kuratowski spaces, $f : X \to Y$ is continuous if for all $A \subseteq X$ it is true that $f[\sigma_X(A)] \subseteq \sigma_Y(f[A])$. Show this is equivalent to continuity in topology. That is, if $(X,\, \tau_X)$ and $(Y,\, \tau_Y)$ are topological spaces, then $f : X \to Y$ is continuous if and only if for all $A \subseteq X$ it is true that $f[\text{Cl}_{\tau_X}(A)] \subseteq \text{Cl}_{\tau_Y}(f[A])$. [Hint: We proved $f : X \to Y$ is continuous if and only if for all closed $\mathcal{D} \subseteq Y$, the pre-image $f^{-1}[\mathcal{D}]$ is closed. Use this definition.]

Solution. As a consequence of $\sigma(A \cup B) = \sigma(A) \cup \sigma(B)$ we have that if $A \subseteq B$, then $\sigma(A) \subseteq \sigma(B)$. This is because, given $A \subseteq B$, we obtain:

$$\sigma(B) = \sigma(A \cup B) = \sigma(A) \cup \sigma(B) \tag{26}$$

so $\sigma(A) \subseteq \sigma(B)$. The set τ_σ is indeed a topology. Firstly, $\emptyset \in \tau_\sigma$. Since $X \subseteq \sigma(X)$, and since $\sigma(X) \subseteq X$, we have that $\sigma(X) = X$, so $\emptyset = X \setminus X$ is an element of τ_σ. Similarly, $X \in \tau_\sigma$ since $\sigma(\emptyset) = \emptyset$, and hence $X = X \setminus \emptyset$ is an element of τ_σ. Let $\mathcal{U}, \mathcal{V} \in \tau_\sigma$. Then $\mathcal{U} = X \setminus \mathcal{C}$ and $\mathcal{V} = X \setminus \mathcal{D}$ for sets $\mathcal{C}$ and $\mathcal{D}$ such that $\sigma(\mathcal{C}) = \mathcal{C}$ and $\sigma(\mathcal{D}) = \mathcal{D}$. But since σ is a Kuratowski closure operator:

$$\sigma(\mathcal{C} \cup \mathcal{D}) = \sigma(\mathcal{C}) \cup \sigma(\mathcal{D}) = \mathcal{C} \cup \mathcal{D} \tag{27}$$

And hence $X \setminus (\mathcal{C} \cup \mathcal{D})$ is an element of τ_σ. But by the De Morgan law:

$$X \setminus (\mathcal{C} \cup \mathcal{D}) = (X \setminus \mathcal{C}) \cap (X \setminus \mathcal{D}) = \mathcal{U} \cap \mathcal{V} \tag{28}$$

so τ_σ is closed under the intersection of two elements. Lastly, let $\mathcal{O} \subseteq \tau_\sigma$. If $\mathcal{O}$ is empty, then $\bigcup \mathcal{O} = \emptyset$, and we've already shown that $\emptyset \in \tau_\sigma$. Suppose $\mathcal{O}$ is non-empty. Then for all $\mathcal{U} \in \mathcal{O}$ there is a $\mathcal{C} \subseteq X$ such that $\mathcal{C} = \sigma(\mathcal{C})$ and $\mathcal{U} = X \setminus \mathcal{C}$. But then:

$$X \setminus \bigcup_{\mathcal{U} \in \mathcal{O}} \mathcal{U} = \bigcap_{\mathcal{U} \in \mathcal{O}} (X \setminus \mathcal{U}) = \bigcap_{\substack{\mathcal{C} = X \setminus \mathcal{U} \\ \mathcal{U} \in \mathcal{O}}} \mathcal{C} \tag{29}$$

To show that $\bigcup \mathcal{O} \in \tau_\sigma$ we must show that:

$$\sigma\Big(\bigcap_{\substack{\mathcal{C} = X \setminus \mathcal{U} \\ \mathcal{U} \in \mathcal{O}}} \mathcal{C} \Big) = \bigcap_{\substack{\mathcal{C} = X \setminus \mathcal{U} \\ \mathcal{U} \in \mathcal{O}}} \mathcal{C} \tag{30}$$

For simplicity, let Δ be the set of all $X \setminus \mathcal{U}$, $\mathcal{U} \in \mathcal{O}$. We have $\bigcap \Delta \subseteq \sigma\big(\bigcap \Delta\big)$ by the property of σ. We must show this reverses. Given any $\mathcal{C} \in \Delta$, by the definition of intersection, we have $\bigcap \Delta \subseteq \mathcal{C}$. But then $\sigma\big(\bigcap \Delta\big) \subseteq \sigma(\mathcal{C})$. But $\sigma(\mathcal{C}) = \mathcal{C}$ for all $\mathcal{C} \in \Delta$. Hence:

$$\sigma\Big(\bigcap \Delta\Big) \subseteq \bigcap_{\mathcal{C} \in \Delta} \mathcal{C} = \bigcap \Delta \tag{31}$$

so $\sigma\big(\bigcap \Delta\big) = \bigcap \Delta$. Hence $X \setminus \bigcap \Delta = \bigcup \mathcal{O}$ is an element of τ_σ. All four criterion are satisfied, so τ_σ is a topology.

Now to prove $f : X \to Y$ is continuous if and only if for all $A \subseteq X$ we have $f\big[\mathrm{Cl}_{\tau_X}(A)\big] \subseteq \mathrm{Cl}_{\tau_Y}\big(f[A]\big)$. Suppose f is continuous. Since $\mathrm{Cl}_{\tau_Y}\big(f[A]\big)$ is closed and f is continuous, $f^{-1}\Big[\mathrm{Cl}_{\tau_Y}\big(f[A]\big)\Big]$ is closed. But $f[A] \subseteq \mathrm{Cl}_{\tau_Y}\big(f[A]\big)$, so:

$$A \subseteq f^{-1}\big[f[A]\big] \subseteq f^{-1}\Big[\mathrm{Cl}_{\tau_Y}\big(f[A]\big)\Big] \tag{32}$$

Therefore:

$$\mathrm{Cl}_{\tau_X}(A) \subseteq \mathrm{Cl}_{\tau_X}\bigg(f^{-1}\Big[\mathrm{Cl}_{\tau_Y}\big(f[A]\big)\Big]\bigg) = f^{-1}\Big[\mathrm{Cl}_{\tau_Y}\big(f[A]\big)\Big] \tag{33}$$

where this last equality comes from the fact that $f^{-1}\Big[\mathrm{Cl}_{\tau_Y}\big(f[A]\big)\Big]$ is closed, since f is continuous, so it is is own closure. From this, we conclude:

$$f\big[\mathrm{Cl}_{\tau_X}(A)\big] \subseteq \mathrm{Cl}_{\tau_Y}\big(f[A]\big) \tag{34}$$

Now, suppose for all $A \subseteq X$ we have that $f\big[\mathrm{Cl}_{\tau_X}(A)\big] \subseteq \mathrm{Cl}_{\tau_Y}\big(f[A]\big)$. Let $\mathcal{D} \subseteq Y$ be closed. Let $\mathcal{C} = f^{-1}[\mathcal{D}]$. We must prove $\mathcal{C}$ is closed. That is, we must prove $\mathrm{Cl}_{\tau_X}(\mathcal{C}) = \mathcal{C}$. It is automatic that $\mathcal{C} \subseteq \mathrm{Cl}_{\tau_X}(\mathcal{C})$, so we must prove $\mathrm{Cl}_{\tau_X}(\mathcal{C}) \subseteq \mathcal{C}$. But:

$$f\big[\mathrm{Cl}_{\tau_X}(\mathcal{C})\big] \subseteq \mathrm{Cl}_{\tau_Y}\big(f[\mathcal{C}]\big) = \mathrm{Cl}_{\tau_Y}\Big(f\big[f^{-1}[\mathcal{D}]\big]\Big) \subseteq \mathrm{Cl}_{\tau_Y}(\mathcal{D}) = \mathcal{D} \tag{35}$$

where we've used the fact that $\mathcal{D}$ is closed, and some of the basic laws of images and pre-images. But then:

$$\mathrm{Cl}_{\tau_X}(\mathcal{C}) \subseteq f^{-1}\Big[f\left[\mathrm{Cl}_{\tau_X}(\mathcal{C})\right]\Big] \subseteq f^{-1}[\mathcal{D}] = \mathcal{C} \tag{36}$$

so $\mathrm{Cl}_{\tau_X}(\mathcal{C}) \subseteq \mathcal{C}$, and hence $\mathcal{C}$ is closed. Thus, f is continuous. □