Point-Set Topology: Lecture 6

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1 More on Open and Closed

Sets are not doors. If $A \subseteq X$ is a subset of a metric space $(X,\, d)$ it does not need to be true that A is either closed or open. A can be open, A can be closed, A can be neither open nor closed, and A can be both open and closed.

Example 1.1 If $(X,\, d)$ is any metric space, then both $\emptyset$ and X are open and closed, simultaneously. ■

Example 1.2 In the real line $\mathbb{R}$ with the standard metric $d(x,\, y) = |x - y|$ there are no proper non-empty subsets that are both open and closed. Open sets are formed by open intervals $(a,\, b)$ and the union of such sets. Closed sets are the complements of these sets. Examples include closed intervals $[a,\, b]$, single points $\{\, x\,\}$, and finite subsets of the real line. There are many other examples not of this form. The rationals $\mathbb{Q}$ are neither closed nor open. It is not open since $(a,\, b)$ always contains irrational numbers for $a < b$, and hence can't be a subset of $\mathbb{Q}$. It is not closed since every irrational can be approximated by a sequence of rationals. ■

Example 1.3 Let X be a non-empty set and let d be the discrete metric:

$$d(x,\, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases} \tag{1}$$

Then *every* subset of X is open, and since the complements of open sets are closed, every set is also closed. To see that every set is open, note that $\{\, x\,\}$ is open for all $x \in X$. Let $r = \frac{1}{2}$. Then $y \in B_r^{(X,\, d)}(x)$ means $d(x,\, y) < \frac{1}{2}$. But the only element that does this is x, so $\{\, x\,\}$ is open. Given $A \subseteq X$ we can write:

$$A = \bigcup_{x \in A} \{\, x\,\} \tag{2}$$

A being the union of open sets, and is thus open. ■

Theorem 1.1. *If $(X,\, d)$ is a metric space, then $\mathcal{U} \subseteq X$ is open if and only if for every convergent sequence $a : \mathbb{N} \to X$ with limit $x \in \mathcal{U}$ it is true that there is an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n > N$ it is true that $a_n \in \mathcal{U}$.*

Proof. Suppose $\mathcal{U}$ is open and $a : \mathbb{N} \to X$ converges to $x \in \mathcal{U}$. Since $x \in \mathcal{U}$ and $\mathcal{U}$ is open, there is an $\varepsilon > 0$ such that $B_{\varepsilon}^{(X,\,d)}(x) \subseteq \mathcal{U}$. But $a_n \to x$ and hence there is an $N \in \mathbb{N}$ such that $n \in \mathbb{N}$ and $n > N$ implies $d(x,\,a_n) < \varepsilon$. Thus $n > N$ implies $a_n \in B_{\varepsilon}^{(X,\,d)}(x)$, so $a_n \in \mathcal{U}$. Now, suppose for every convergent sequence $a : \mathbb{N} \to X$ with limit $x \in \mathcal{U}$ there is an $N \in \mathbb{N}$ with $n \in \mathbb{N}$ and $n > N$ implying $a_n \in \mathcal{U}$. Suppose $\mathcal{U}$ is not open. Then there is an $x \in \mathcal{U}$ such that for all $\varepsilon > 0$ there is a point $y \in X$ such that $d(x,\,y) < \varepsilon$ but $y \notin \mathcal{U}$. In particular, for all $n \in \mathbb{N}$ there is an a_n such that $d(x,\,a_n) < \frac{1}{n+1}$ but $a_n \notin \mathcal{U}$. Then $a_n \to x$. But $x \in \mathcal{U}$ and hence there is an $N \in \mathbb{N}$ such that $n \in \mathbb{N}$ and $n > N$ implies $a_n \in \mathcal{U}$. But $a_n \notin \mathcal{U}$, a contradiction. Therefore, $\mathcal{U}$ is open. □

2 Subspaces

Many new metric spaces are formed by considering metric spaces we've already constructed and examining subsets of these spaces.

Theorem 2.1. *If $(X,\,d)$ is a metric space, if $A \subseteq X$, and if d_A is the restriction of d to $A \times A$, then $(A,\,d_A)$ is a metric space.*

Proof. For all $a, b, c \in A$ it is true that $a, b, c \in X$ since $A \subseteq X$. Hence $d_A(a,\,b) = d(a,\,b)$ and so d_A is positive-definite, symmetric, and satisfies the triangle inequality. Thus, $(A,\,d_A)$ is a metric space. □

Definition 2.1 (**Metric Subspace**) A metric subspace of a metric space $(X,\,d)$ is a metric space $(A,\,d_A)$ where A is a subset $A \subseteq X$ and d_A is the restriction of d to $A \times A$. ■

Example 2.1 Equip $\mathbb{R}^2$ with the Euclidean metric and define $\mathbb{S}^1$ to be the set of all $\mathbf{x} \in \mathbb{R}^2$ such that $||\mathbf{x}||_2 = 1$. This is the unit circle in the plane. We turn this into a metric space by equipping it with the subspace metric from $\mathbb{R}^2$. This is the standard metric on $\mathbb{S}^1$. This is shown in Fig. 1. ■

Example 2.2 The unit sphere is defined similarly as a subspace of $\mathbb{R}^3$. It is the set of all points $\mathbf{x} \in \mathbb{R}^3$ such that $||\mathbf{x}||_2 = 1$. In general the n dimensional sphere $\mathbb{S}^n$ is the subset of all points in $\mathbb{R}^{n+1}$ such that $||\mathbf{x}||_2 = 1$. ■

If $(X,\,d)$ is a metric space, if $(A,\,d_A)$ is a metric subspace, and if $\mathcal{U} \subseteq A$ is open with respect to d_A, it need not be true that $\mathcal{U}$ is open with respect to d. For example, the real line can be viewed as a subset of the plane by identifying $x \in \mathbb{R}$ with $(x,\,0) \in \mathbb{R}^2$. An open subset of the real line is a interval, but open subsets in the plane are blobs (*two dimensional*). What is true is the following.

Theorem 2.2. *If $(X,\,d)$ is a metric space, if $A \subseteq X$ and $(A,\,d_A)$ is a metric subspace, then $\mathcal{U} \subseteq A$ is open with respect to d_A if and only if there is an open subset $\mathcal{V} \subseteq X$ with respect to d such that $\mathcal{U} = \mathcal{V} \cap A$.*

Proof. If $\mathcal{U}$ is open in A, then for all $x \in \mathcal{U}$ there is an $r_x > 0$ such that $B_{r_x}^{(A,\,d_A)}(x) \subseteq \mathcal{U}$. Define $\mathcal{O}$ by:

$$\mathcal{O} = \{\, B_{r_x}^{(X,\,d)}(x) \mid x \in \mathcal{U} \,\} \tag{3}$$

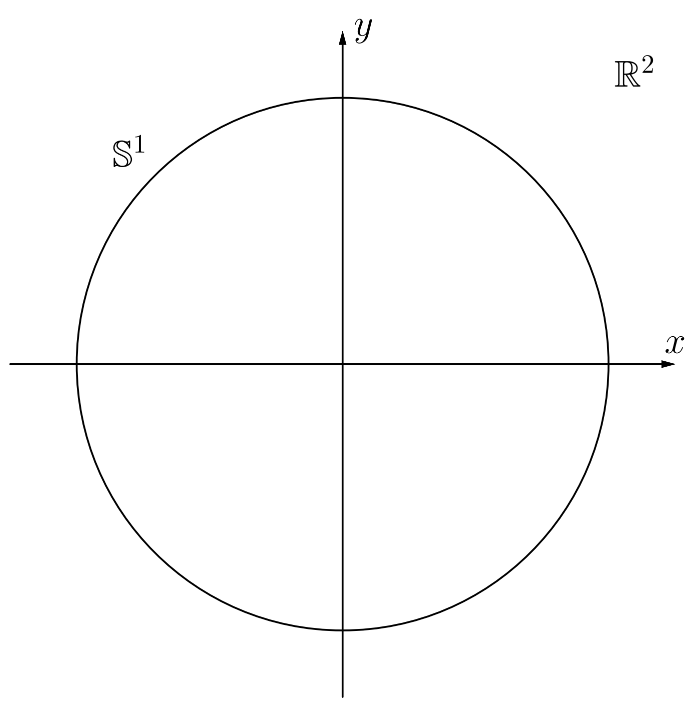

Figure 1: $\mathbb{S}^1$ as a Subspace of $\mathbb{R}^2$

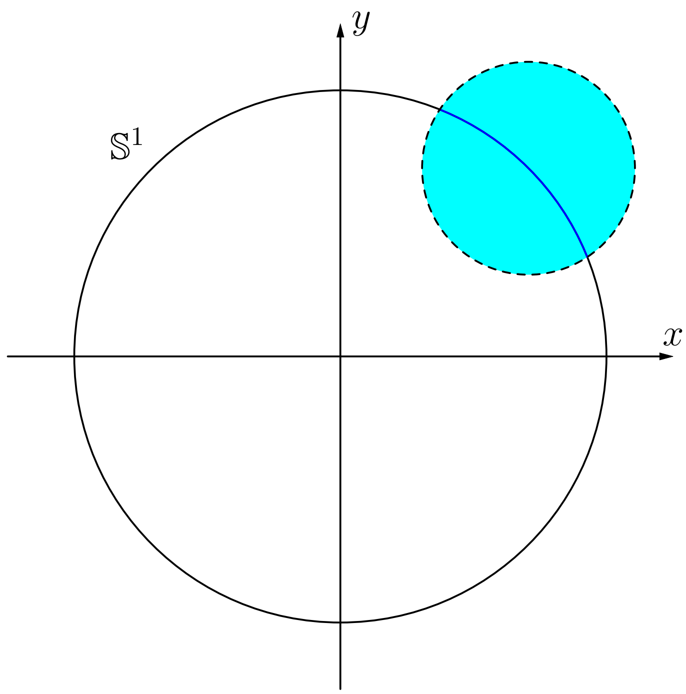

Figure 2: An Open Subset of $\mathbb{S}^1$

Note that these open balls are open balls in (X, d), not (A, d_A). Let $\mathcal{V} = \bigcup \mathcal{O}$. Since $\mathcal{V}$ is the union of open balls, it is open in X. If $x \in \mathcal{U}$, then $x \in A$ since $\mathcal{U} \subseteq A$, and also $x \in \mathcal{V}$ since $x \in B_{r_x}^{(X,\, d)}(x)$ which is a subset of $\bigcup \mathcal{O}$. Hence $x \in \mathcal{V} \cap A$, and therefore $\mathcal{U} \subseteq \mathcal{V} \cap A$. Next, suppose $x \in \mathcal{V} \cap A$. Then $x \in \mathcal{V}$ and hence there is a $y \in \mathcal{U}$ such that $x \in B_{r_y}^{(X,\, d)}(y)$. But $y \in A$, and hence $d(x, y) = d_A(x, y)$, and so $x \in B_{r_y}^{(A,\, d_A)}(y)$, and thus $x \in \mathcal{U}$. That is, $\mathcal{V} \cap A \subseteq \mathcal{U}$, and therefore $\mathcal{U} = \mathcal{V} \cap A$. Now suppose $\mathcal{U} = \mathcal{V} \cap A$ where $\mathcal{V}$ is open in X. Let $x \in \mathcal{U}$. Then $x \in \mathcal{V} \cap A$, and hence $x \in \mathcal{V}$. But $\mathcal{V}$ is open, and therefore there is an $r > 0$ such that $B_r^{(X,\, d)}(x) \subseteq \mathcal{V}$. Now given $y \in B_r^{(A,\, d_A)}(x)$, by definition $y \in A$ and $d(x, y) = d_A(x, y)$. Therefore $y \in \mathcal{V} \cap A$. But $\mathcal{U} = \mathcal{V} \cap A$, so $y \in \mathcal{U}$. Since $y \in B_r^{(A,\, d_A)}(x)$ is arbitrary, we have that $B_r^{(A,\, d_A)}(x) \subseteq \mathcal{U}$, so $\mathcal{U}$ is open in A. □

Example 2.3 An *open arc* in the circle, an arc that does not include the endpoints, is open. See Fig. 2. ■

3 Continuity

Like most branches of mathematics, there's a notion of *identical* metric spaces, or *equivalent* metric spaces. The only thing that defines a metric is the set X and the metric d. If we relabel the points in X, forming a new set Y, but preserve the distances, then we haven't really changed the metric space. For example, we can define on $\mathbb{R}^2$ the metric $d_{\mathbb{R}^2}(\mathbf{x},\,\mathbf{y})$ to be the Euclidean metric, $d_{\mathbb{R}^2}(\mathbf{x},\,\mathbf{y}) = ||\mathbf{x}-\mathbf{y}||_2$. We can relabel $(x,\,y) = x+iy$ and call this the *complex plane*, denote it $\mathbb{C}$, and given $z = x_0 + ix_1$ and $w = y_0 + iy_1$, we could define $d_{\mathbb{C}}(z,\,w) = |z-w|$ where $|\cdot|$ is the complex absolute value. This is no different from the Euclidean metric, all we've done is relabel everything. We use this to motivate isometries.

Definition 3.1 (**Metric Space Isometry**) A metric space isometry from a metric space $(X,\,d_X)$ to a metric space $(Y,\,d_Y)$ is a function $f:X\to Y$ such that for all $x_0, x_1 \in X$ it is true that:

$$d_X(x_0,\,x_1) = d_Y\big(f(x_0),\,f(x_1)\big) \tag{4}$$

That is, f preserves the metrics. ■

Example 3.1 In Euclidean geometry one often ponders isometries of the plane to itself. It is a classic result that all isometries $f:\mathbb{R}^2\to\mathbb{R}^2$ are combinations of *translations*, *reflections*, and *rotations* (The so-called *glide-reflections* are reflections followed by translations). A translation is a function $f:\mathbb{R}^2\to\mathbb{R}^2$ defined by $f(\mathbf{x}) = \mathbf{x}+\mathbf{y}$ for some fixed $\mathbf{y}\in\mathbb{R}^2$. Reflections are functions that *flip* bases. For example, a reflection about the y axis is the function $f(\mathbf{x}) = (-x_0,\,x_1)$. A reflection about the x axis is of the form $f(\mathbf{x}) = (x_0,\,-x_1)$. Reflections about other lines through the origin can be defined similarly. Lastly, rotation by an angle θ is the function defined by:

$$f(\mathbf{x}) = \begin{bmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}x_0\\ x_1\end{bmatrix} \tag{5}$$

$$= (\cos(\theta)x_0 + \sin(\theta)x_1,\,-\sin(\theta)x_0+\cos(\theta)x_1) \tag{6}$$

An isometry $f:\mathbb{R}^2\to\mathbb{R}^2$ is a combination of these operations. ■

Theorem 3.1. *If $(X,\,d_X)$ and $(Y,\,d_Y)$ are metric spaces, and if $f:X\to Y$ is a metric space isometry, then f is injective.*

Proof. For if $x_0 \neq x_1$, then $d_X(x_0,\,x_1) > 0$, and thus $d_Y\big(f(x_0),\,f(x_1)\big) = d_X(x_0,\,x_1) > 0$, so $f(x_0)\neq f(x_1)$. □

Definition 3.2 (**Global Metric Space Isometry**) A global metric space isometry from a metric space $(X,\,d_X)$ to a metric space $(Y,\,d_Y)$ is a bijective metric space isometry $f:X\to Y$. ■

Global isometries mean $(X,\,d_X)$ and $(Y,\,d_Y)$ are, in a sense, the *same* metric space. Just like $\mathbb{R}^2$ and $\mathbb{C}$ can be thought of as the same, so can X and Y by identifying $x\in X$ with $f(x)\in Y$ and $y\in Y$ with $f^{-1}(y)\in X$.

There is a much weaker notion than isometries for metric spaces, and this notion is far more useful.

Definition 3.3 (**Continuous Function Between Metric Spaces**) A continuous function from a metric space $(X,\, d_X)$ to a metric space $(Y,\, d_Y)$ is a function $f:X\rightarrow Y$ such that for every convergent sequence $a:\mathbb{N}\rightarrow X$ with limit $x\in X$ it is true that $f(a_n)\rightarrow f(x)$. ■

Using notation from calculus, this says that:

$$\lim_{n\to\infty} f(a_n) = f\big(\lim_{n\to\infty} a_n\big) = f(x) \tag{7}$$

This reads nicely as *the image of a convergent sequence is a convergent sequence.* Similarly you could say *f maps convergent sequences to convergent sequences.*

The simplest functions from calculus (constants and the identity) are always continuous, regardless of the metrics.

Theorem 3.2. *If $(X,\, d_X)$ and $(Y,\, d_Y)$ are metric spaces, and if $f:X\rightarrow Y$ is a constant function, then f is continuous.*

Proof. For let $a:\mathbb{N}\rightarrow X$ be a convergent sequence. Since f is a contant function there is a $y\in Y$ such that $f(x)=y$ for all $x\in X$. But then $f(a_n)=y$ for all $n\in\mathbb{N}$, meaning the image of the sequence under f is a constant sequence, which is in turn a convergent sequence. Hence f is continuous. □

Theorem 3.3. *If $(X,\, d)$ is a metric space and if $\mathrm{id}_X:X\rightarrow X$ is the identity function, $\mathrm{id}_X(x)=x$, then id_X is continuous.*

Proof. For if $a:\mathbb{N}\rightarrow X$ is a convergent sequence, then $\mathrm{id}_X(a_n)=a_n$, which is still a convergent sequence. So id_X is continuous. □

Theorem 3.4. *If $(X,\, d_X)$ and $(Y,\, d_Y)$ are metric spaces, and if $f:X\rightarrow Y$ is an isometry, then f is continuous.*

Proof. For let $a:\mathbb{N}\rightarrow X$ be a convergent sequence with limit $x\in X$. Then given $\varepsilon>0$ there is an $N\in\mathbb{N}$ such that $n\in\mathbb{N}$ and $n>N$ implies $d_X(a_n,\, x)<\varepsilon$. But then for all $n>N$ we have:

$$d_Y\big(f(a_n),\, f(x)\big) = d_X(a_n,\, x) < \varepsilon \tag{8}$$

and hence $f(a_n)\rightarrow f(x)$. That is, f is continuous. □

This theorem does not reverse. Since isometries must be injective, any non-injective constant function (i.e., the co-domain has more than two points) would be continuous but not an isometry.

Theorem 3.5. *If $(X,\, d_X)$ and $(Y,\, d_Y)$ are metric spaces, and if $f:X\rightarrow Y$ is a function, then f is continuous if and only if for all open $\mathcal{V}\subseteq Y$ it is true that $f^{-1}[\mathcal{V}]\subseteq X$ is open.*

Proof. Suppose f is continuous and $\mathcal{V} \subseteq Y$ is open. Let $\mathcal{U} = f^{-1}[\mathcal{V}]$. Suppose $a : \mathbb{N} \to X$ is such that $a_n \to x$ with $x \in \mathcal{U}$. Since $a_n \to x$ and f is continuous, $f(a_n) \to f(x)$. But since $x \in f^{-1}[\mathcal{V}]$ it is true that $f(x) \in \mathcal{V}$. But $\mathcal{V}$ is open, so there exists an $N \in \mathbb{N}$ such that $n \in \mathbb{N}$ and $n > N$ implies $f(a_n) \in \mathcal{V}$. But then for all $n > N$, $a_n \in \mathcal{U} = f^{-1}[\mathcal{V}]$, so $\mathcal{U}$ is open. Now, suppose $f : X \to Y$ is such that for all open $\mathcal{V} \subseteq Y$ it is true that $f^{-1}[\mathcal{V}]$ is open. Let $a : \mathbb{N} \to X$ be a convergent sequence that converges to $x \in X$. Suppose $f(a_n)$ does not converge to $f(x)$. Then there exists an $\varepsilon > 0$ such that for all $N \in \mathbb{N}$ there is an $n \in \mathbb{N}$ with $n > N$ but $d_Y\big(f(x),\, f(a_n)\big) \geq \varepsilon$. But $B_{\varepsilon}^{(Y,\, d_Y)}\big(f(x)\big)$ is open, so by assumption the pre-image is open. Letting $\mathcal{V} = B_{\varepsilon}^{(Y,\, d_Y)}\big(f(x)\big)$, we have that $f^{-1}[\mathcal{V}]$ is open. But $x \in f^{-1}[\mathcal{V}]$ since $f(x) \in \mathcal{V}$. Since $f^{-1}[\mathcal{V}]$ is open there is an $N \in \mathbb{N}$ such that $n \in \mathbb{N}$ and $n > N$ implies $a_n \in f^{-1}[\mathcal{V}]$. But then $f(a_n) \in \mathcal{V}$ for all $n > N$ which is a contradiction. Therefore, $f(a_n) \to f(x)$ and f is continuous. $\square$