Point-Set Topology: Lecture 8

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1 Theorems on Compactness

Theorem 1.1. *If* $(X,\,d)$ *is a metric space and* $a:\mathbb{N}\rightarrow\mathbb{R}$ *is a convergent sequence, then* a *is a Cauchy sequence.*

Proof. Since $a:\mathbb{N}\rightarrow X$ converges, there is an $x\in X$ such that $a_n\rightarrow x$. Let $\varepsilon>0$. Since $a_n\rightarrow x$ there is an $N\in\mathbb{N}$ such that $n>N$ implies $d(x,\,a_n)<\frac{\varepsilon}{2}$. But then $n,m>N$ implies:

$$d(a_m,\,a_n)\leq d(x,\,a_m)+d(x,\,a_n)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon \tag{1}$$

and therefore a is a Cauchy sequence. □

Without completeness, a metric space $(X,\,d)$ can have non-convergent Cauchy sequences. But given a Cauchy sequence with a convergent subsequence, the entire sequence must then convergent. The intuition is that a Cauchy sequence is a sequence where all of the points start to get really close together as the indices increase. Since there is a convergent subsequence, there is some point $x\in X$ where *some* of the points in the sequence start to get really close to. But since all of the points get close together at higher and higher indices, all of the points must also get closer to x, and hence the entire sequence converges to x. Let's prove this.

Theorem 1.2. *If* $(X,\,d)$ *is a metric space, if* $a:\mathbb{N}\rightarrow X$ *is a Cauchy sequence, and if* a_k *is a convergent subsequence, then* a *is a convergent sequence.*

Proof. Since a_k is a convergent sequence, there is an $x\in X$ such that $a_{k_n}\rightarrow x$. Let $\varepsilon>0$. Since $a_{k_n}\rightarrow x$, there is an $N_0\in\mathbb{N}$ such that $n>N_0$ implies $d(x,\,a_{k_n})<\frac{\varepsilon}{2}$. Since a is a Cauchy sequence there is an $N_1\in\mathbb{N}$ such that $n,m>N_1$ implies $d(a_m,\,a_n)<\frac{\varepsilon}{2}$. Let $N=\max(k_{N_0},\,N_1)$. Then since k is strictly increasing, $m>N$ implies $k_m>N_0$ and $k_m>N_1$. But then for all $n,m>N$:

$$d(x,\,a_n)\leq d(a_n,\,a_{k_m})+d(x,\,a_{k_m})<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon \tag{2}$$

and therefore $a_n\rightarrow x$. That is, a is a convergent sequence. □

Completeness and closedness are related for metric space. Given a complete metric space (X, d), the only complete subspaces are the closed ones. In particular, since $(\mathbb{R}, |\cdot|)$, the standard metric on the real line, is complete, we see that the open unit interval is *not* complete. The sequence $a : \mathbb{N} \to (0, 1)$ defined by $a_n = \frac{1}{n+1}$ is a Cauchy sequence, but it does not converge. We want to say it "converges" to zero, but zero is not an element of this subspace.

Theorem 1.3. *If (X, d) is a complete metric space, and if $A \subseteq X$, then (A, d_A) is a complete metric space if and only if A is closed.*

Proof. Suppose (A, d_A) is a complete metric space and let $a : \mathbb{N} \to A$ be a sequence that converges in X. But convergent sequences are Cauchy sequences, and (A, d_A) is complete, and therefore Cauchy sequences converge. But then the limit of a is contained in A, and therefore A is closed. Now, suppose $A \subseteq X$ is closed. Let $a : \mathbb{N} \to A$ be a Cauchy sequence. Then, since $A \subseteq X$, $a : \mathbb{N} \to X$ is a Cauchy sequence in X. But (X, d) is complete, and therefore a converges. That is, there is an $x \in X$ such that $a_n \to x$. But A is closed and therefore contains all of its limit points, so $x \in A$. Hence Cauchy sequences in A converge in A, and therefore (A, d_A) is complete. □

This theorem is the baby version of the same idea for compactness.

Theorem 1.4. *If (X, d) is a compact metric space, and $A \subseteq X$, then (A, d_A) is compact if and only if A is closed.*

Proof. Suppose (A, d_A) is compact and $x \in X$ a limit point of A. Then there is a sequence $a : \mathbb{N} \to A$ such that a_n converges to x in X. But (A, d_A) is compact, so there is a convergent subsequence a_k with limit in A. But limits are unique, so $a_{k_n} \to x$, and therefore $x \in A$. Now suppose A is closed. Let $a : \mathbb{N} \to A$ be a sequence. Then, since $A \subseteq X$, $a : \mathbb{N} \to X$ is a sequence in X. But (X, d) is compact, so there is a convergent subsequence a_k with limit $x \in X$. But A is closed and hence contains all of its limit points, and therefore $x \in A$. But then a_k is a convergent subsequence of a in A. Therefore, (A, d_A) is compact. □

And lastly, it should be noted that compactness is far stronger than completeness. Let's prove this.

Theorem 1.5. *If (X, d) is a compact metric space, then (X, d) is complete.*

Proof. Let $a : \mathbb{N} \to X$ be a Cauchy sequence. Since (X, d) is compact, there is a convergent subsequence a_k. But a Cauchy sequence with a convergent subsequence is convergent, and therefore a is convergent. Thus, (X, d) is complete. □

Theorem 1.6 (**Heine-Borel Theorem**). *If $A \subseteq \mathbb{R}^N$ and $d : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is the standard Euclidean metric, $d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||_2$, then (A, d_A) is compact if and only if A is closed and bounded.*

Proof. Suppose $(A,\, d_A)$ is compact. We have proved that compact subspaces of any metric space are closed, so in particular A is a closed subset of $\mathbb{R}^N$. Suppose A is not bounded. Then for all $n \in \mathbb{N}$ there is an $a_n \in A$ with $||a_n||_2 > n$, otherwise A is bounded. The sequence $a : \mathbb{N} \to A$ can be chosen so that $||a_m||_2 < ||a_n||_2$ whenever $m < n$, while diverging off to infinity, and hence contains no convergent subsequences. This contradicts the assumption that $(A,\, d_A)$ is compact. Hence, A is bounded. Now, suppose $(A,\, d_A)$ is closed and bounded. Let $\mathbf{x} : \mathbb{N} \to A$ be any sequence. Denote $\mathbf{x}_n \in A$ via the tuple:

$$\mathbf{x}_n = \big(x_n^0,\, x_n^1,\, \dots,\, x_n^{N-1}\big) \tag{3}$$

The sequence $x^0 : \mathbb{N} \to \mathbb{R}$ defined by setting x_n^0 equal to the zeroth component of $\mathbf{x}_n$ is bounded since A is bounded. By the Bolzano-Weierstrass theorem there is a convergent subsequence x_k^0. That is, there is some real number $r_0 \in \mathbb{R}$ such that $x_{k_n}^0 \to r_0$. But then x_k^1 is a (not necessarily convergent) subsequence of the sequence $x^1 : \mathbb{N} \to \mathbb{R}$, the sequence defined by setting x_n^1 equal to the first component of $\mathbf{x}_n$. But then x_k^1 is a bounded sequence of real numbers and hence by the Bolzano-Weierstrass theorem, there is a convergent subsequence $x_{k_{k'}}^1$. That is, there is some $r_1 \in \mathbb{R}$ such that $x_{k_{k'_n}}^1 \to r_1$. But $x_{k_{k'}}^0$ is a subsequence of x_k^0, and hence $x_{k_{k'}}^0$ is a subsequence of a convergent subsequence. But subsequences of convergent sequences converge and they converge to the same limit. That is, we now have that $x_{k_{k'}}^0$ and $x_{k_{k'}}^1$ are convergent sequences converging to r_0 and r_1, respectively. Continuing inductively, we obtain subseqence k'', k''', up to $k^{(N-1)}$ such that $x_{k\dots k^{(N-1)}}^m$ is a convergent sequence for all $m \in \mathbb{Z}_N$ where $k \dots k^{(N-1)}$ denotes the sequence obtain by repeated composition:

$$k \dots k^{N-1}(n) = k\Bigg(k'\Big(\cdots\big(k^{(N-1)}(n)\big)\Big)\cdots\Bigg) \tag{4}$$

But then $\mathbf{x}_{k\dots k^{(N-1)}}$ is a sequence in A that converges to the $\mathbf{y} \in \mathbb{R}^N$ defined by $\mathbf{y} = (r_0,\, r_1,\, \dots,\, r_{N-1})$. But $A \subseteq \mathbb{R}^N$ is closed and hence contains all of its limit points, so $\mathbf{y} \in A$. That is, $\mathbf{x}$ has a convergent subsequence. Therefore $(A,\, d_A)$ is compact. □

Do not attempt to apply this result to general metric spaces. The Heine-Borel theorem is specific to Euclidean spaces with the Euclidean metric (or topologically equivalent metrics such as the Manhattan and maximum metrics). The discrete metric on any set is bounded. In particular, the discrete metric on $\mathbb{R}$ is bounded. Moreover, since every subset of a discrete metric space is open, every subset of a discrete metric space is closed (being the complement of an open set). Any infinite subset of $\mathbb{R}$ with the discrete metric is *not* compact. In the discrete metric space $(X,\, d)$ a sequence $a : \mathbb{N} \to X$ converges to $x \in X$ if and only if there is some $N \in \mathbb{N}$ such that $n > N$ implies $a_n = x$. That is, convergent sequences are eventually constant. (To see this, apply the definition of convergence to the positive number $\varepsilon = \frac{1}{2}$). So given an infinite subset A of $\mathbb{R}$ we can find an injective sequence $a : \mathbb{N} \to A$. A is indeed closed and bounded, but a has no convergent subsequence. So A is not compact with the discrete metric.