Point-Set Topology: Lecture 9

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1 The Equivalence of Compactness Theorem

The generalization of the Heine-Borel theorem for general metric spaces needs stronger notions than just *closed* and *bounded*. If we replace *closed* with *complete* and *bounded* with *totally bounded*, we get the generalized Heine-Borel theorem.

Theorem 1.1 (Generalized Heine-Borel Theorem). *If $(X,\,d)$ is a metric space, then it is compact if and only if it is complete and totally bounded.*

Proof. We have already proven that compact implies complete. Now let's show compact implies totally bounded. Suppose not. Then there is an $\varepsilon > 0$ such that no matter what finite collection of points $a_0,\,\dots,\,a_n$ you pick, there is another point a_{n+1} where $a_{n+1} \notin B_{\varepsilon}^{(X,\,d)}(a_k)$ for all $0 \leq k \leq n$. Inductively this defines a sequence $a : \mathbb{N} \to X$. But $(X,\,d)$ is compact, so there is a convergent subsequence a_k. But convergent sequences are Cauchy sequences, and therefore there there is an $N \in \mathbb{N}$ such that $m, n > N$ implies $d(a_{k_n},\,a_{k_m}) < \varepsilon$. But by construction $d(a_{k_n},\,a_{k_m}) \geq \varepsilon$ for all distinct $n, m \in \mathbb{N}$, a contradiction. So $(X,\,d)$ is totally bounded. Now, suppose $(X,\,d)$ is complete and totally bounded. Let $a : \mathbb{N} \to X$ be any sequence. Since $(X,\,d)$ is totally bounded, there are finitely many points $b_0,\,\dots,\,b_N$ such that the open balls $B_1^{(X,\,d)}(b_k)$ completely cover X. Since there are infinitely integers and only finitely many open balls, there must be a point b_k such that infinitely many $n \in \mathbb{N}$ are such that $a_n \in B_1^{(X,\,d)}(b_k)$. Let $k_0 \in \mathbb{N}$ be such a value with $a_{k_0} \in B_0^{(X,\,d)}(b_k)$. Again by total boundedness, we can cover $B_1^{(X,\,d)}(b_k)$ with finitely many open balls $B_{1/2}^{(X,\,d)}(c_\ell)$ with points $c_\ell \in B_1^{(X,\,d)}(b_k)$. Since infinitely many integers $n \in \mathbb{N}$ are such that $a_n \in B_1^{(X,\,d)}(b_k)$ and there are only finitely many balls covering this set, one of these open balls must again be such that there are infinitely many integers $n \in \mathbb{N}$ with $a_n \in B_{1/2}^{(X,\,d)}(c_\ell)$. Let k_1 be such that $k_1 > k_0$ and $a_{k_1} \in B_{1/2}^{(X,\,d)}(c_\ell)$. Inductively we get a subsequence a_k with the property that $a_{k_{n+1}}$ is contained inside the ball of radius $\frac{1}{n+1}$ centered at a_{k_n}. This sequence is Cauchy since $d(a_{k_m},\,a_{k_n})$ is bounded by $\frac{1}{N+1}$ where $N = \min(m,\,n)$. But $(X,\,d)$ is complete, so this sequence converges. Hence a has a convergent subsequence. $\square$

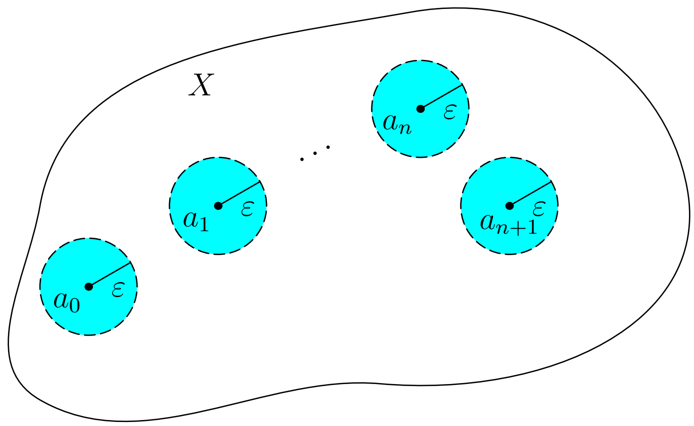

Figure 1: A Non-Totally Bounded Metric Space

We can describe this theorem pictorially. Not totally bounded means there is an $\varepsilon > 0$ such that no finite set of open balls of radius ε completely cover X. We get Fig. 1. This sequence $a : \mathbb{N} \to X$ has the property that for all $n \neq m$ we have $d(a_n, a_m) \geq \varepsilon$, so a cannot possible have any convergent subsequences, violating compactness. The latter direction, that totally bounded and complete implies compact, is pictorial as well. We cover our metric space in open balls of radius 1. This is possible since the space is totally bounded. We look at our sequence $a : \mathbb{N} \to X$. There must be an open ball with infinitely many a_n contained inside it since there are infinitely many integers and only finitely many open balls. This is shown in Fig. 2. Our sequence is the points in red, and the red ball is an open ball that contains infinitely many of the a_n. Note, there could be two such balls, or three. In this figure there's only one. We then zoom in on this open ball and cover it in finitely many balls of radius $\frac{1}{2}$ (Fig. 3). We continue and obtain a sequence of nested open balls of radius $\frac{1}{n+1}$ and a subsequence of points a_{k_n} such that a_{k_n} lies in the n^{th} ball. The subsequence must be Cauchy since the distance between two points is bounded by the diameter (twice the radius) of the $\frac{1}{n+1}$ balls, which tends to zero. Since the space is complete, this subsequence converges.

Compactness is a topological property, but we've yet to describe it in terms of open sets. We use the generalized Heine-Borel theorem to get one step closer to this, but first we need a definition.

Definition 1.1 (**Open Cover of a Metric Space**) An open cover of a metric space (X, d) is a subset $\mathcal{O} \subseteq \tau_d$, where τ_d is the metric topology, such that $\bigcup \mathcal{O} = X$. That is, for all $x \in X$, there is an open set $\mathcal{U} \in \mathcal{O}$ such that $x \in \mathcal{U}$. ■

Open covers are the topological tool needed to define compactness. The slogan for compactness is that *every open cover has a finite subcover*, a phrase that

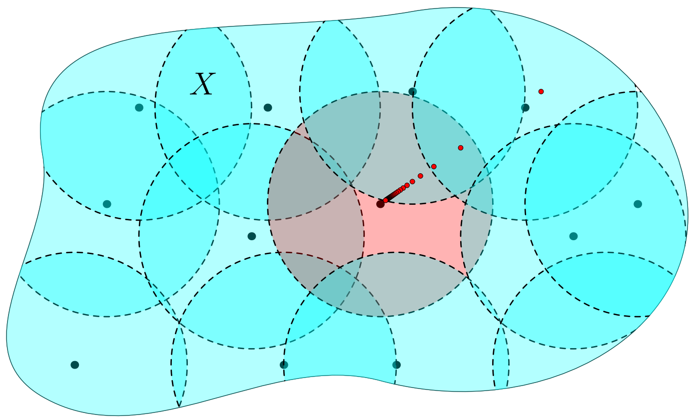

Figure 2: A Sequence in a Totally Bounded Metric Space

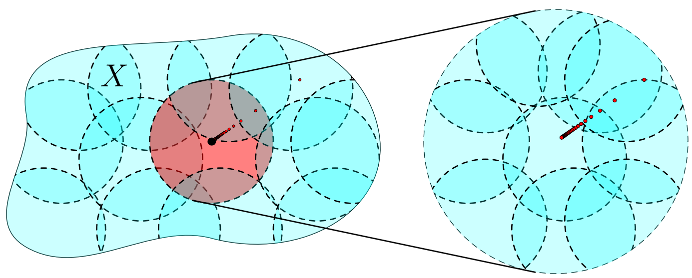

Figure 3: Zooming in on a Totally Bounded Metric Space

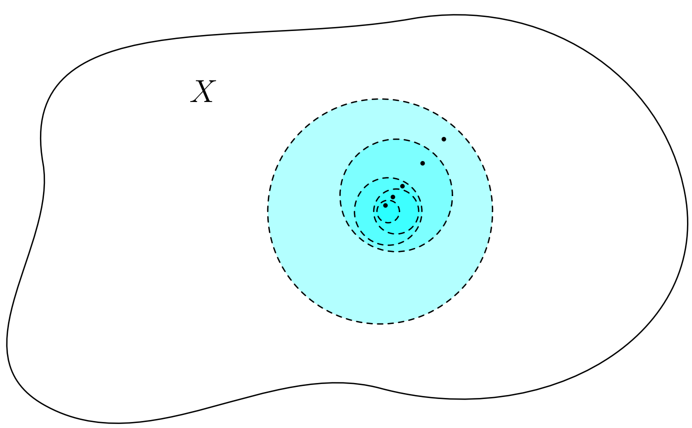

Figure 4: Convergent Subsequence in on a Totally Bounded Metric Space

is dependent solely on the notion of open sets. To prove the equivalence of compactness theorem, we first need a lemma.

Theorem 1.2 (**Lebesgue's Number Lemma**). *If* $(X,\, d)$ *is compact, and if* $\mathcal{O}$ *is an open cover of* $(X,\, d)$*, then there is a* $\delta > 0$ *such that for all* $x \in X$ *there is a* $\mathcal{U} \in \mathcal{O}$ *such that* $B_{\delta}^{(X,\, d)}(x) \subseteq \mathcal{U}$*.*

Proof. Suppose not. Then for all $n \in \mathbb{N}$ there is an a_n such that $B_{\frac{1}{n+1}}^{(X,\, d)}(a_n)$ is not contained entirely inside of $\mathcal{U}$ for any $\mathcal{U} \in \mathcal{O}$. But $(X,\, d)$ is compact, so there is a convergent subsequence a_k. Let $x \in X$ be the limit, $a_{k_n} \to x$. Since $\mathcal{O}$ is an open cover, there is a $\mathcal{U}$ with $x \in \mathcal{U}$. But $\mathcal{U}$ is open, so there is an $\varepsilon > 0$ such that $B_{\varepsilon}^{(X,\, d)}(x) \subseteq \mathcal{U}$. But since $a_{k_n} \to x$ there is an $N_0 \in \mathbb{N}$ such that $k_n > N_0$ implies $d(x,\, a_{k_n}) < \frac{\varepsilon}{2}$. Let $N_1 \in \mathbb{N}$ be such that $N_1 + 1 > \frac{2}{\varepsilon}$ and let $N = \max(N_0,\, N_1) + 1$. Then $y \in B_{\frac{1}{N+1}}^{(X,\, d)}(a_N)$ implies:

$$d(x,\, y) \leq d(x,\, a_N) + d(a_N,\, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \tag{1}$$

and hence $y \in B_{\varepsilon}^{(X,\, d)}(x)$. That is, $B_{\frac{1}{N+1}}^{(X,\, d)}(a_N) \subseteq B_{\varepsilon}^{(X,\, d)}(x)$. But the ε ball around x is contained inside of $\mathcal{U}$, so $B_{\frac{1}{N+1}}^{(X,\, d)}(a_N) \subseteq \mathcal{U}$, which is a contradiction, completing the proof. □

Theorem 1.3 (**The Equivalence of Compactness Theorem**). *If* $(X,\, d)$ *is a metric space, then* $(X,\, d)$ *is compact if and only if for every open cover* $\mathcal{O}$ *of* $(X,\, d)$ *there exists a finite open cover* $\Delta \subseteq \mathcal{O}$*.*

Proof. If $(X,\, d)$ is compact and $\mathcal{O}$ is an open cover, then by the Lebesgue number lemma there is a $\delta > 0$ such that for all $x \in X$ there is a $\mathcal{U}_x$ such that $B_{\delta}^{(X,\, d)}(x) \subseteq \mathcal{U}_x$. Since $(X,\, d)$ is compact, it is totally bounded, and hence there are finitely many points $x_0,\, \dots,\, x_N$ such that the δ balls centered at x_n cover X. But then the set:

$$\Delta = \{\, \mathcal{U}_{x_0},\, \dots,\, \mathcal{U}_{x_N} \,\} \tag{2}$$

is a finite open cover that is a subset of $\mathcal{O}$. In the other direction, suppose $(X,\, d)$ is such that every open cover $\mathcal{O}$ contains a finite subset $\Delta \subseteq \mathcal{O}$ that is also an open cover of $(X,\, d)$. Given $\varepsilon > 0$, create the set $\mathcal{O}$ by:

$$\mathcal{O} = \{\, B_{\varepsilon}^{(X,\, d)}(x) \mid x \in X \,\} \tag{3}$$

$\mathcal{O}$ is an open cover of $(X,\, d)$, and hence there is a finite open cover $\Delta \subseteq \mathcal{O}$. But then $(X,\, d)$ can be covered by finitely many balls of radius ε, so $(X,\, d)$ is totally bounded. Next, suppose $(X,\, d)$ is not complete. There there is a Cauchy sequence $a : \mathbb{N} \to X$ that does not converge. Then for all $x \in X$ there is a $\varepsilon_x > 0$ such that for all $N \in \mathbb{N}$ there exists $n > N$ with $d(x,\, a_n) \geq \varepsilon$. Let $\mathcal{O}$ be defined by:

$$\mathcal{O} = \{\, B_{\varepsilon_x/2}^{(X,\, d)}(x) \mid x \in X \,\} \tag{4}$$

$\mathcal{O}$ is an open cover of $(X,\, d)$, so there is a finite open cover $\Delta \subseteq \mathcal{O}$. Let $x_0,\, \dots, x_N$ be the finite set of points such that:

$$X = \bigcup_{n=0}^{N} B_{\varepsilon_n/2}^{(X,\, d)}(x_n) \tag{5}$$

where $\varepsilon_n = \varepsilon_{x_n}$. Let $\varepsilon = \min(\varepsilon_0,\, \dots,\, \varepsilon_N)$. Since $a : \mathbb{N} \to X$ is a Cauchy sequence, there is an $N \in \mathbb{N}$ such that $n, m > N$ implies $d(a_m,\, a_n) < \varepsilon/2$. Since the open balls of radius $\varepsilon_k/2$ centered at x_k cover the metric space, there is some x_k, $0 \leq k \leq N$, such that $d(a_{N+1},\, x_k) < \varepsilon_k/2$. But then for all $n > N$ we have:

$$d(x_k,\, a_n) \leq d(x_k,\, a_{N+1}) + d(a_n,\, a_{N+1}) < \frac{\varepsilon_n}{2} + \frac{\varepsilon}{2} \leq \varepsilon_n \tag{6}$$

Which is a contradiction since for all $N \in \mathbb{N}$ there is an $n > N$ such that $d(x_k,\, a_n) \geq \varepsilon_n$ by the definition of ε_n. Therefore $(X,\, d)$ is complete. Since $(X,\, d)$ is complete and totally bounded, by the generalized Heine-Borel theorem it is compact. $\square$