Point-Set Topology: Lecture 10

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1 Topological Spaces

Definition 1.1 (**Topology on a Set**) A topology on a set X is a subset $\tau \subseteq \mathcal{P}(X)$ such that:

1. $\emptyset \in \tau$
2. $X \in \tau$
3. For every $\mathcal{O} \subseteq \tau$ it is true that $\bigcup \mathcal{O} \in \tau$
4. For all $\mathcal{U}, \mathcal{V} \in \tau$ it is true that $\mathcal{U} \cap \mathcal{V} \in \tau$.

That is, τ contains the empty set and the whole set, it is closed under arbitrary unions, and closed under the intersection of two elements. ■

Theorem 1.1. *If X is a set, if τ is a topology on X, and if $\mathcal{O} \subseteq \tau$ is finite, then $\bigcap \mathcal{O} \in \tau$.*

Proof. We prove by induction on the size of $\mathcal{O}$. The base case is true by the definition of a topology. Suppose the claim is true for $n \in \mathbb{N}$ and let $\mathcal{O} \subseteq \tau$ have $n+1$ elements. Label them $\mathcal{U}_0, \dots, \mathcal{U}_n$. Then, by induction, the set $\mathcal{V}$ defined by:

$$\mathcal{V} = \bigcap_{k=0}^{n-1} \mathcal{U}_k \tag{1}$$

is in τ. But then $\bigcap \mathcal{O} = \mathcal{V} \cap \mathcal{U}_n$, the intersection of two elements of τ, which is an element of τ. Hence, $\bigcap \mathcal{O} \in \tau$. □

Definition 1.2 (**Topological Space**) A topological space is an ordered pair (X, τ) where X is a set and τ is a topology on X. ■

Example 1.1 If X is a set, then $\mathcal{P}(X)$, the power set of X, is a topology on X. The power set is trivially closed under arbitrary unions and finite intersections, and moreover $\emptyset \in \mathcal{P}(X)$ and $X \in \mathcal{P}(X)$. This is the *discrete topology* on X. ■

Example 1.2 If X is a set, then the set $\tau = \{\,\emptyset,\, X\,\}$ is a topology on X. This has several names, the *chaotic topology*, the *trivial topology*, and the *indiscrete topology*. ■

Example 1.3 Take $X = \{0, 1, 2\}$ and $\tau = \{\emptyset, \{0\}, \{0, 1, 2\}\}$. The set τ is a topology on X. The sets are all nested since $\emptyset \subseteq \{0\} \subseteq \{0, 1, 2\}$, so it is closed under unions and intersections. ■

In metric spaces we used the metric d to define openness. Here, we use the topology.

Definition 1.3 (**Open Set in a Topological Space**) An open set in a topological space (X, τ) is an element $\mathcal{U} \in \tau$. ■

In the metric setting we were able to use sequences to define limit points and closed sets. This gave us a theorem that closed sets are just the complements of open sets. Since we lack a metric, we take this and use it to *define* closed sets in a topological space.

Definition 1.4 (**Closed Set in a Topological Space**) A closed set in a topological space (X, τ) is a set $\mathcal{C} \subseteq X$ such that $X \setminus \mathcal{C}$ is open. That is, a set $\mathcal{C}$ such that $X \setminus \mathcal{C} \in \tau$. ■

Theorem 1.2. *If (X, τ) is a topological space, then $\mathcal{U} \subseteq X$ is open if and only if $X \setminus \mathcal{U}$ is closed.*

Proof. Suppose $\mathcal{U}$ is open. Then by definition, $X \setminus \mathcal{U}$ is closed. Now, suppose $X \setminus \mathcal{U}$ is closed. Then by the definition of closed sets, $X \setminus (X \setminus \mathcal{U})$ is open. But by the double complement law, $\mathcal{U} = X \setminus (X \setminus U)$, and hence $\mathcal{U}$ is open. □

Example 1.4 Do not confuse *open* for *not-closed* and do not confuse *not-open* for *closed*. These are distinct notions. It is possible to be open, closed, both, or neither. Let X be a set and let $\tau = \mathcal{P}(X)$, the discrete topology on X. Then *every* subset is open, and by the previous theorem, *every* subset is closed. Now let $\tau = \{\emptyset, X\}$. Then every non-empty proper subset is *not* open, and hence every non-empty proper subset is *not* closed. These two examples show it is possible for $\mathcal{U} \subseteq X$ to be both open and closed, and for $\mathcal{V} \subseteq X$ to be neither open nor closed. It completely depends on the topology τ. ■

Topological spaces are direct generalizations of metric spaces. Every metric space is also a topological space.

Theorem 1.3. *If (X, d) is a metric space, and if τ_d is the metric topology, then (X, τ_d) is a topological space.*

Proof. We have proven that $\emptyset$ and X are metrically open, so $\emptyset, X \in \tau_d$. Moreover, the intersection of two open sets in a metric space is open, as is the union of arbitrarily many open sets. That is, τ_d is a topology on X, and (X, τ_d) is a topological space. □

Example 1.5 The standard topology on the real line $\mathbb{R}$, the metric topology induced by the metric $d(x, y) = |x - y|$, denoted $\tau_{\mathbb{R}}$, is such that every proper non-empty subset $\mathcal{U} \subseteq \mathbb{R}$ is either open, closed, or neither, but *not* both. That is, the only subsets of $\mathbb{R}$ that are both open and closed are $\emptyset$ and $\mathbb{R}$. This isn't too easy to show, it is essentially the statement that the real line is *connected*, but we don't have the vocabulary to prove such a claim yet. Still, it is worth knowing this property when trying to intuitively understand open and closed sets. ■

A natural question is whether or not all topological spaces arise from metrics. This is false. The indiscrete topology on a set X containing at least two points can't come from a metric. For suppose X is a set with $a, b \in X$ and $a \neq b$. Suppose d is any metric. Setting $\varepsilon = d(a,\, b)/2$, the open ball around a of radius ε is a metrically open subset that contains a but does not contain b. But in the indiscrete topology $\tau = \{\,\emptyset,\, X\,\}$, the only open sets are the empty set (which $B_\varepsilon^{(X,\, d)}(a)$ is not empty since $\varepsilon > 0$ and hence $a \in B_\varepsilon^{(X,\, d)}(a)$) and the whole space X (and $B_\varepsilon^{(X,\, d)}(a) \neq X$ since $b \notin B_\varepsilon^{(X,\, d)}(a)$). So the indiscrete topology cannot come from a metric.

One of the problems with the indiscrete topology is that it lacks the *Hausdorff property*. In Felix Hausdorff's original definition of topological spaces he required points in the space to be able to be separated by disjoint open sets. That is, given $x, y \in X$ with $x \neq y$, Hausdorff required there to be open sets $\mathcal{U}$ and $\mathcal{V}$ such that $x \in \mathcal{U}$, $y \in \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V} = \emptyset$. We take Hausdorff's property and use it to define *Hausdorff topological spaces* (also see Fig. 1).

Definition 1.5 (**Hausdorff Topological Space**) A Hausdorff topological space is a topological space $(X,\, \tau)$ such that for all $x, y \in X$, $x \neq y$, there exists $\mathcal{U}, \mathcal{V} \in \tau$ such that $x \in \mathcal{U}$, $y \in \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V} = \emptyset$. ■

There are reasons we take the more general definition as *the* definition of a topological space. There are certain operations that can be performed on a topological space (such as glueing points together) that can take a Hausdorff space $(X,\, \tau)$ and transform it into a non-Hausdroff space (but it'll still be a topological space). Also many non-Hausdorff topological spaces have found their way into the mathematical world with applications to physics and geometry. In algebraic geometry, the *Zariski topology* is non-Hausdorff, and in general relativity, the space of light rays in a spacetime can be non-Hausdorff, depending on the topology of the spacetime.

Metric spaces have the Hausdorff property, but since we've moved on to topology, it is better to speak of *metrizable spaces*. These are topological spaces where the topology τ comes from some metric d.

Definition 1.6 (**Metrizable Topological Space**) A metrizable topological space is a topological space $(X,\, \tau)$ such that there exists a metric d on X such that $\tau = \tau_d$, where τ_d is the metric topology from d. ■

Every metric space we've examined is an example of a metrizable topological space. In particular, the real line, Cartesian or complex plane, Euclidean space, Paris plane, London plane, discrete metric spaces, all of that. We've also found a non-metrizable space, the indiscrete topology on a set containing at least

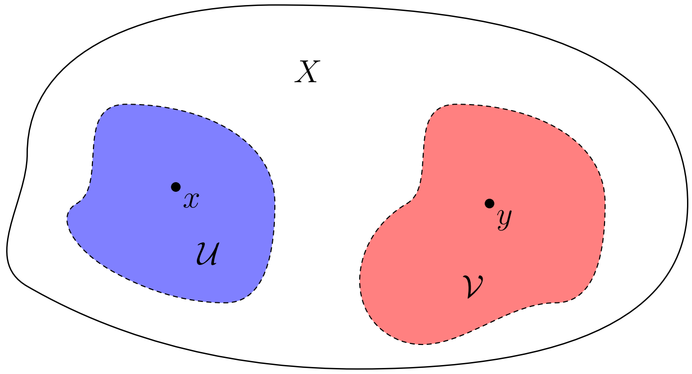

Figure 1: The Hausdorff Condition

two points. Some of the central theorems of point-set topology tell us when a topological space is metrizable. Hausdorff alone is not enough. We will see plenty of Hausdorff spaces that can not come from a metric. The converse is true, however. Every metrizable space is Hausdorff.

Theorem 1.4. *If $(X,\,\tau)$ is a metrizable topological space, then it is Hausdorff.*

Proof. Since $(X,\,\tau)$ is metrizable, there is a metric d such that $\tau = \tau_d$, where τ_d is the metric topology from d. Let $x, y \in X$ with $x \neq y$. To prove $(X,\,\tau)$ is Hausdorff we must find disjoint open sets $\mathcal{U}$ and $\mathcal{V}$ such that $x \in \mathcal{U}$ and $y \in \mathcal{V}$. Let $\varepsilon = d(x,\,y)/2$ and define $\mathcal{U} = B_{\varepsilon}^{(X,\,d)}(x)$ and $\mathcal{V} = B_{\varepsilon}^{(X,\,d)}(y)$. Since $x \neq y$ and d is a metric, it is true that $d(x,\,y) > 0$ and therefore ε is positive. But then $\mathcal{U}$ and $\mathcal{V}$ are open sets with $x \in \mathcal{U}$ and $y \in \mathcal{V}$, since open balls are open. Suppose $z \in \mathcal{U} \cap \mathcal{V}$. Then:

$$d(x,\,y) \leq d(x,\,z) + d(z,\,y) < \varepsilon + \varepsilon = d(x,\,y) \tag{2}$$

And thus $d(x,\,y) < d(x,\,y)$, a contradiction. So $z \notin \mathcal{U} \cap \mathcal{V}$, and therefore $\mathcal{U} \cap \mathcal{V} = \emptyset$. Thus, $(X,\,\tau)$ is Hausdorff. $\square$

See Fig. 2 for the idea behind the proof. With this we can rigorously prove that the indiscrete topology on a set X that has at least two distinct points $a, b \in X$ cannot possibly come from a metric. This is done by observing that, if $\tau = \{\,\emptyset,\,X\,\}$, then $(X,\,\tau)$ is *non-Hausdorff*, and since it is non-Hausdorff, it can't be metrizable since metrizable topological spaces are Hausdorff. Given $a \in X$, the only open set in τ that contains a is X (since $a \notin \emptyset$). But $b \in X$ as well, so there are not two open sets $\mathcal{U}$ and $\mathcal{V}$ with $a \in \mathcal{U}$, $b \in \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V} = \emptyset$. That is, $(X,\,\tau)$ is not Hausdorff, meaning it is not metrizable. Topological spaces are far more general than metric spaces.

Example 1.6 Let's intuitively try to imagine what the indiscrete topology on $\mathbb{R}$ *looks* like. The indiscrete topology $\tau = \{\,\emptyset,\,\mathbb{R}\,\}$ says the only non-empty open set is the entire real line $\mathbb{R}$. That is, all of the points in the real line are

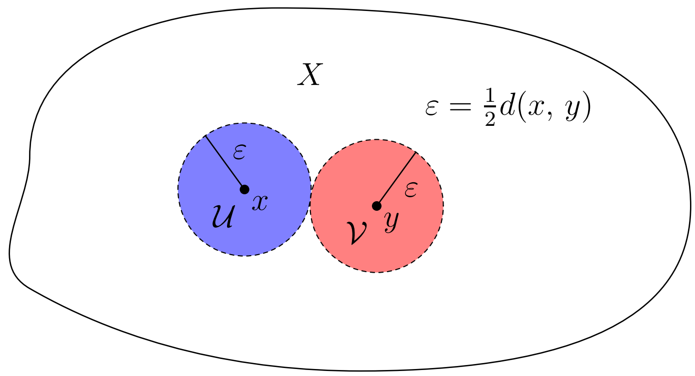

Figure 2: Metrizable Implies Hausdorff

very very close to each other, essentially indistinguishable. Topologically it is impossible to tell two points in the line apart. This is hard to imagine, and your brain probably pictures a single point. But the space $(\mathbb{R},\, \tau)$ is not the same as a single point, the set $\mathbb{R}$ still has infinitely many points. This is hard to imagine, and fortunately such bizarre topological spaces like $(\mathbb{R},\, \tau)$ rarely find their way into applications. Spaces like this usually serve as counterexamples to claims (such as the claim that every topological space comes from a metric. The topological space $(\mathbb{R},\, \tau)$ is a counterexample). ■

Example 1.7 Now ponder the discrete topology on $\mathbb{R}$, $\tau = \mathcal{P}(\mathbb{R})$. Here, *every* subset is open, so in particular given $x, y \in \mathbb{R}$, the sets $\{\, x \,\}$ and $\{\, y \,\}$ are open. The way to picture this space is as a bunch of scattered points with empty space between them. ■

There is a weaker notion than Hausdorff that has found it's way into physics and geometry, the notion of a *Fréchet topological space*.

Definition 1.7 (**Fréchet Topological Space**) A Fréchet topological space is a topological space $(X,\, \tau)$ such that for all $x, y \in X$, $x \neq y$, there exists $\mathcal{U}, \mathcal{V} \in \tau$ such that $x \in \mathcal{U}$, $y \in \mathcal{V}$, and $x \notin \mathcal{V}$, $y \notin \mathcal{U}$. ■

See Fig. 2 for a visual. This modifies the Hausdorff condition. Instead of requiring $\mathcal{U}$ and $\mathcal{V}$ to be disjoint (as in the Hausdroff case), we only require $\mathcal{U}$ to include x and not y, and $\mathcal{V}$ to include y and not x. Every Hausdorff space is, in particular, Fréchet.

Theorem 1.5. *If $(X,\, \tau)$ is a Hausdorff topological space, then it is a Fréchet topological space.*

Proof. Let $x, y \in X$, $x \neq y$. Since $(X,\, \tau)$ is Hausdorff, there exists open sets $\mathcal{U}, \mathcal{V} \in \tau$ such that $x \in \mathcal{U}$, $y \in \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V} = \emptyset$. But since $x \in \mathcal{U}$, it must be true that $x \notin \mathcal{V}$ since $\mathcal{U} \cap \mathcal{V} = \emptyset$. But since $y \in \mathcal{V}$, it must be true that $y \notin \mathcal{U}$ since $\mathcal{U} \cap \mathcal{V} = \emptyset$. So $\mathcal{U}$ and $\mathcal{V}$ are such that $x \in \mathcal{U}$, $y \in \mathcal{V}$, and $x \notin \mathcal{V}$, $y \notin \mathcal{U}$. So $(X,\, \tau)$ is a Fréchet topological space. □

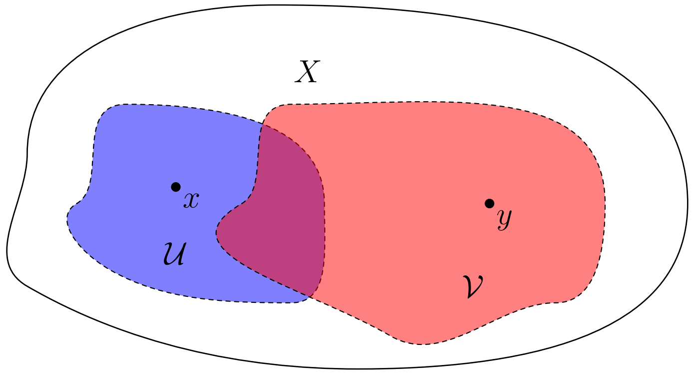

Figure 3: The Fréchet Condition

A word of warning. There are three types of spaces that are called *Fréchet* spaces. Maurice René Fréchet was a very prolific mathematician. In the study of topological vector spaces one studies *Fréchet spaces*. In general topology one studies *Fréchet topological spaces* and *Fréchet-Urysohn topological spaces*. Fréchet also invented the idea of metric spaces, but fortunately the mathematical community has universally adopted the term *metric space*, rather than name another type of space after Fréchet. To reduce confusion, Fréchet topological spaces are often called T_1 spaces, and Hausdorff topological spaces are sometimes called T_2. It is far more common to just say *Hausdorff*, and this T_n notation seems to be mostly historical.

Example 1.8 (**Finite Complement Topology**) Not every Fréchet topological space is Hausdorff. The space of light rays in a spacetime, as alluded to earlier, need not be Hausdorff, but light rays always form a Fréchet topological space. A simpler example is the *finite complement topology* on the real line. Define a set $\mathcal{U}\subseteq\mathbb{R}$ to be open if and only if $\mathbb{R}\setminus\mathcal{U}$ is a finite set, or if $\mathcal{U}=\emptyset$. The collection τ of all such sets is a topology on $\mathbb{R}$. Since $\mathbb{R}\setminus\mathbb{R}=\emptyset$, which is finite, we see that $\mathbb{R}\in\tau$. The empty set was intentionally included, so $\emptyset\in\tau$. If $\mathcal{U},\mathcal{V}\in\tau$, then:

$$\mathbb{R}\setminus(\mathcal{U}\cap\mathcal{V})=(\mathbb{R}\setminus\mathcal{U})\cup(\mathbb{R}\setminus\mathcal{U}) \tag{3}$$

by the De Morgan law. Since this is the union of two finite sets, it is finite, and hence τ is closed under the intersection of two elements. Lastly, if $\mathcal{O}\subseteq\tau$, then:

$$\mathbb{R}\setminus\bigcup_{\mathcal{U}\in\mathcal{O}}\mathcal{U}=\bigcap_{\mathcal{U}\in\mathcal{O}}\mathbb{R}\setminus\mathcal{U} \tag{4}$$

If $\mathcal{O}$ is empty, then this intersection is $\mathbb{R}$, and $\mathbb{R}$ is open. If $\mathcal{O}$ is non-empty, then there is some $\mathcal{V}\in\mathcal{O}$. But then, by the definition of intersection:

$$\bigcap_{\mathcal{U}\in\mathcal{O}}\mathbb{R}\setminus\mathcal{U}\subseteq\mathbb{R}\setminus\mathcal{V} \tag{5}$$

which is a finite set, so the intersection is a subset of a finite set, and is therefore finite itself. This shows $\bigcup \mathcal{O} \in \tau$, so τ is a topology. $(\mathbb{R}, \tau)$ is not Hausdorff. Given any non-empty $\mathcal{U}, \mathcal{V} \in \tau$, since $\mathbb{R} \setminus \mathcal{U}$ and $\mathbb{R} \setminus \mathcal{V}$ are finite, and since $\mathbb{R}$ is infinite, $\mathcal{U} \cap \mathcal{V}$ is non-empty. $(\mathbb{R}, \tau)$ is a Fréchet topological space, however. Given $x \neq y$, let $\mathcal{U} = \mathbb{R} \setminus \{ y \}$ and $\mathcal{V} = \mathbb{R} \setminus \{ x \}$. Then $x \in \mathcal{U}$ since $x \neq y$, but $y \notin \mathcal{U}$. Also, $y \in \mathcal{V}$, but $x \notin \mathcal{V}$. But $\mathcal{U}$ and $\mathcal{V}$ are open since their complements contain one point, and are hence finite. This shows $(\mathbb{R}, \tau)$ is a Fréchet topological space. ■

Example 1.9 (**Countable Complement Topology**) Define τ on $\mathbb{R}$ to be the set of all $\mathcal{U} \subseteq \mathbb{R}$ such that $\mathbb{R} \setminus \mathcal{U}$ is countable (that is, finite or countably infinite), or $\mathcal{U}$ is the empty set. This is a topology on $\mathbb{R}$. The empty set and $\mathbb{R}$ are elements of τ since $\mathbb{R} \setminus \mathbb{R} = \emptyset$ is countable, and $\emptyset$ was deliberately included. Given $\mathcal{U}, \mathcal{V} \in \tau$, the intersection is also included since:

$$\mathbb{R} \setminus (\mathcal{U} \cap \mathcal{V}) = (\mathbb{R} \setminus \mathcal{U}) \cup (\mathbb{R} \setminus \mathcal{V}) \tag{6}$$

which is the union of countable sets, and is therefore countable. By a similar argument to the finite complement topology, the union of any collection $\mathcal{O} \subseteq \tau$ is also an element of τ. The space $(\mathbb{R}, \tau)$ is not Hausdorff, but it is a Fréchet topological space. It is not Hausdorff since for any non-empty $\mathcal{U}, \mathcal{V} \in \tau$, the complements $\mathbb{R} \setminus \mathcal{U}$ and $\mathbb{R} \setminus \mathcal{V}$ are countable. The real numbers are uncountable, meaning there must be an element common to both $\mathcal{U}$ and $\mathcal{V}$, showing $(\mathbb{R}, \tau)$ cannot be Hausdorff. To show it is Fréchet we use the same construction as before. Given $x, y \in \mathbb{R}$, $x \neq y$, we define $\mathcal{U} = \mathbb{R} \setminus \{ y \}$ and $\mathcal{V} = \mathbb{R} \setminus \{ x \}$, showing $(\mathbb{R}, \tau)$ has the Fréchet property. ■

The construction used in both of these examples relies on the fact that $\mathbb{R} \setminus \{ x \}$ is an *open* subset for any real number $x \in \mathbb{R}$ in both the finite and countable complement topologies. To phrase this differently, the proof requires that $\{ x \}$ is a *closed* set. This is what our intuition tells us. Singleton sets should always be closed. In a metric space (X, d), given $x \in X$, the set $\{ x \}$ is indeed closed since the only sequence $a : \mathbb{N} \to \{ x \}$ is the constant sequence $a_n = x$, and this does indeed converge to x, showing that $\{ x \}$ has all of its limit points. This property does not exist for all topological spaces. The real line $\mathbb{R}$ with the indiscrete topology $\tau = \{ \emptyset, \mathbb{R} \}$ lacks this feature. The *only* closed sets are $\emptyset$ and $\mathbb{R}$, so in particular, given $x \in \mathbb{R}$, it is *not* true that $\{ x \}$ is closed. Fréchet topological spaces (and hence Hausdorff topological spaces) do not lack this feature.

Theorem 1.6. *If (X, τ) is a Fréchet topological space, and if $x \in X$, then the set $\{ x \}$ is closed.*

Proof. For all $y \in X$, $y \neq x$, there is an open set $\mathcal{U}_y \in \tau$ such that $x \notin \mathcal{U}_y$ and $y \in \mathcal{U}_y$. Define $\mathcal{O}$ by:

$$\mathcal{O} = \{ \mathcal{U}_y \in \tau \mid y \in X \text{ and } y \neq x \} \tag{7}$$

Since $\mathcal{O}$ is a collection of open sets, $\bigcup \mathcal{O}$ is open. Since for all $y \in X$ with $y \neq x$ the set $\mathcal{U}_y$ is contained in $\mathcal{O}$, we have that $y \in \bigcup \mathcal{O}$. But also for every open set

$\mathcal{U}$ in $\mathcal{O}$, $x \notin \mathcal{U}$ by construction, and hence $x \notin \bigcup \mathcal{O}$. Therefore $\bigcup \mathcal{O} = X \setminus \{ x \}$. But then $\{ x \}$ is the complement of an open set, and is therefore closed. $\square$

The converse of this theorem is true as well, and this is what we used to show that the finite complement and countable complement topologies on $\mathbb{R}$ are Fréchet topological spaces.

Theorem 1.7. *If (X, τ) is a topological space, and if for all $x \in X$ it is true that $\{ x \}$ is closed, then (X, τ) is a Fréchet topological space.*

Proof. For given $x, y \in X$, $x \neq y$, define $\mathcal{U} = X \setminus \{ y \}$ and $\mathcal{V} = X \setminus \{ x \}$. Then $x \in \mathcal{U}$ since $x \neq y$, and $y \in \mathcal{V}$ since $y \neq x$. But also $x \notin \mathcal{V}$ and $y \notin \mathcal{U}$. Since $\{ x \}$ and $\{ y \}$ are closed, by hypothesis, the sets $\mathcal{U}$ and $\mathcal{V}$ are the complements of closed sets and are therefore open. Thus, (X, τ) is a Fréchet topological space. $\square$

Example 1.10 (**Standard Topology on** $\mathbb{R}$) Let $\tau_{\mathbb{R}}$, the *standard topology* on $\mathbb{R}$, be defined as the set of all $\mathcal{U} \subseteq \mathbb{R}$ such that $x \in \mathcal{U}$ implies there is an $\varepsilon > 0$ such that for all $y \in \mathbb{R}$ with $|x - y| < \varepsilon$, it is true that $y \in \mathcal{U}$. Then $(\mathbb{R}, \tau_{\mathbb{R}})$ is a topological space, and moreover it is a *metrizable* topological space since it stems from the standard Euclidean metric $d(x, y) = |x - y|$ defined on $\mathbb{R}$. Since this is metrizable, $(\mathbb{R}, \tau_{\mathbb{R}})$ is a Hausdorff topological space and a Fréchet topological space. $\blacksquare$

Let τ_F denote the finite complement topology on $\mathbb{R}$, τ_C the countable complement topology on $\mathbb{R}$, and $\tau_{\mathbb{R}}$ the standard topology on $\mathbb{R}$. Since a finite set is countable, we instantly have that $\tau_F \subseteq \tau_C$. But also, since $(\mathbb{R}, \tau_{\mathbb{R}})$ is a metrizable, and therefore Hausdorff, and thus a Fréchet topological space, points $\{ x \}$ with $x \in \mathbb{R}$ are closed in $\tau_{\mathbb{R}}$. Since the finite union of closed sets is still closed, we see that all finite subsets of $\mathbb{R}$ are closed in $\tau_{\mathbb{R}}$, and therefore sets whose complement is finite are open in $\tau_{\mathbb{R}}$. But then $\tau_F \subseteq \tau_{\mathbb{R}}$. There is no comparison between τ_C and $\tau_{\mathbb{R}}$. The set $(0,\, 1)$ is open in $\tau_{\mathbb{R}}$ but not τ_C since the complement of $(0,\, 1)$ is $(-\infty,\, 0] \cup [1,\, \infty)$, and this is not countable. Moreover, the set of all irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is open in τ_F since the complement is the rational numbers $\mathbb{Q}$ and this is countable. The irrationals are not open in $\tau_{\mathbb{R}}$ since for any irrational x and for any $\varepsilon > 0$ there is a rational number y with $|x - y| < \varepsilon$. This notion of comparing topologies allows us to generate new topologies. In particular, if we are given a collection of topologies on a set X, we can construct a new topology by looking at the intersection over this collection.

Theorem 1.8. *If X is a set, and if $T \subseteq \mathcal{P}\big(\mathcal{P}(X)\big)$ is a non-empty set such that for all $\tau \in T$ it is true that τ is a topology on X, then $\bigcap T$ is a topology on X.*

Proof. Since for all $\tau \in T$, τ is a topology, we have that $\emptyset \in \tau$ and $X \in \tau$, and since T is non-empty, we conclude that $\emptyset \in \bigcap T$ and $X \in \bigcap T$. If $\mathcal{U}, \mathcal{V} \in \bigcap T$, then for all $\tau \in T$, it is true that $\mathcal{U}, \mathcal{V} \in \tau$. But all $\tau \in T$ are topologies, so $\mathcal{U} \cap \mathcal{V} \in \tau$ for all τ, so $\mathcal{U} \cap \mathcal{V} \in \bigcap T$. If $\mathcal{O} \subseteq \bigcap T$, then for all $\tau \in T$ we have $\mathcal{O} \subseteq \tau$. But then, since τ is a topology, $\bigcup \mathcal{O} \in \tau$. Since this is true for all $\tau \in T$ we have that $\bigcup \mathcal{O} \in \bigcap T$. So $\bigcap T$ is a topology. $\square$

Given a set X, we use this theorem to define the topology *generated* by any subset $B \subseteq \mathcal{P}(X)$. We collect the *smallest* topology that contains B as a subset. There is always a topology that contains B as a subset since $\mathcal{P}(X)$ is a topology. We write T as the set of all topologies τ on X with $B \subseteq \tau$ and then write $\tau(B) = \bigcap T$.

Definition 1.8 (**Generated Topology**) The topology on a set X generated by a collection $B \subseteq \mathcal{P}(X)$ is the set $\tau(B)$ defined by:

$$\tau(B) = \bigcap T \tag{8}$$

where T is the set of all topologies τ on X such that $B \subseteq \tau$. ■

What we are doing is taking a collection of subsets $B \subseteq \mathcal{P}(X)$ that we *want* to be open. However, this collection might not be a topology itself. We may need to add more sets to ensure we have the intersection and union properties that topologies enjoy. $\tau(B)$, the generated topology, does precisely this.